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HERON QUADRILATERALS WITH SIDES IN ARITHMETIC OR GEOMETRIC PROGRESSION

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We study triangles and cyclic quadrilaterals which have rational area and whose sides form geometric or arithmetic progressions. A complete characterisation is given for the infinite family of triangles with sides in arithmetic progression. We show that there are no triangles with sides in geometric progression. We also show that apart from the square there are no cyclic quadrilaterals whose sides form either a geometric or an arithmetic progression. The solution of both quadrilateral cases involves searching for rational points on certain elliptic curves.

1. INTRODUCTION

A recent article [4] treated the problem of finding Heron triangles having sides whose lengths are consecutive integers. In a subsequent article [5], the second author showed how to characterise all such triangles. Indeed, it was shown there how to find all Heron triangles with sides whose lengths form an arithmetic progression. At the same time Beauregard and Suryanarayan published two papers [1] and [2] in which they drew the same conclusion. In this paper, we extend the problem to search for triangles with rational area which have rational sides in geometric progression. We show that no such triangles exist by showing that the problem equates to solving a certain diophantine equation.

In addition we investigate the problem of finding cyclic quadrilaterals of rational area having rational sides in either arithmetic or geometric progression. A trivial example is the square whose sides form both a degenerate arithmetic progression (a common difference of 0) and a degenerate geometric progression (a common ratio of 1). We prove that apart from this example, none exists. In both cases the problem reduces to finding the rational points on an elliptic curve which is then shown to have rank 0.

2. TRIANGLES WITH SIDES IN ARITHMETIC PROGRESSION

For completeness, we mention briefly the case of triangles with rational area having rational sides in arithmetic progression (as described in [5] and [2]). Let rational numbers

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a, b, and c be the sides of a triangle having rational area. Using Heron's formula, we can express the area as

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where s is the semi-perimeter. Since the sides are in arithmetic progression, we may write them as $a = b - d$ and $c = b + d$ where $d < b$. Then the area becomes

$$\frac{b}{4}\sqrt{3\,(b^2 - 4d^2)}$$

and the requirement is that this be rational. Thus we must have $3\,(b^2 - 4d^2)$ a rational square. Setting $x = 2d$, this amounts to asking for the rational solutions to

$$x^2 + 3y^2 = b^2.$$

Dividing through by b^2 (which is non-zero) allows us to write this as $X^2 + 3Y^2 = 1$ whose solutions are easily found by the chord method [7] to be

$$X = \frac{1 - 3t^2}{1 + 3t^2}, \qquad Y = \frac{2t}{1 + 3t^2}$$

From this we obtain

$$d = \frac{(1 - 3t^2)b}{2(1 + 3t^2)}$$

and the area of the triangle is then $3tb^2/2(1 + 3t^2)$. Thus we have

THEOREM 1. *A triangle with rational sides a, b, c in arithmetic progression has rational area if and only if the common difference is $d = b(1 - 3t^2)/2(1 + 3t^2)$ where t is an arbitrary rational number.*

We can carry out a similar analysis in the case where we want the sides and area to be integers. We obtain a similar homogeneous quadratic equation which is now to have integer solutions. The complete set of primitive solutions forms an infinite family giving

$$d = (m^2 - 3n^2)/g,$$
$$b = 2(m^2 + 3n^2)/g$$

where m and n are relatively prime integers and $g = \gcd(\, m^2 - 3n^2, 2mn, m^2 + 3n^2)$. This family includes the obvious $3 - 4 - 5$ right-angled triangle. The details are found in [5] or [1] and the reader is left with the exercise of showing that $d \equiv \pm 1 (\mathrm{mod}\, 12)$ in this case.

3. Triangles with Sides in Geometric Progression

In this section we consider the case of triangles with rational area having sides in geometric progression. If we let the sides be a, ar, ar^2 where $a, r \in \mathbf{Q}$ and $r \neq 0$ then the semiperimeter is $s = a(1 + r + r^2)/2$. As before we use Heron's formula to compute the area; this yields

$$A = \frac{a^2}{4}\sqrt{(1+r+r^2)(-1+r+r^2)(1-r+r^2)(1+r-r^2)}$$

and for this to be rational, we must have

$$(1+r+r^2)(-1+r+r^2)(1-r+r^2)(1+r-r^2) = y^2$$

where $y \in \mathbf{Q}$. Now we set $r = m/n$, where $m, n \in \mathbf{Z}$, $(m, n) = 1$ and this gives the integer equation

$$Y^2 = (n^2+mn+m^2)(-n^2+mn+m^2)(n^2-mn+m^2)(n^2+mn-m^2)$$

where now $Y \in \mathbf{Z}$. Now it is easily checked that the four terms in the above product are pairwise relatively prime. This means that each term is separately a square. In fact it will be sufficient to use the fact that a product of two of the terms is a square. Consequently we examine the equation

$$\begin{aligned} Y'^2 &= (n^2+mn+m^2)(n^2-mn+m^2) \\ &= n^4 + n^2m^2 + m^4 \end{aligned}$$

According to Mordell [6, p.19] the only solution to this equation has $mn = 0$, and since $n \neq 0$ this yields only $r = 0$ in the geometric progression. So we have proved the following:

Theorem 2. *There are no triangles with rational area having rational sides in geometric progression.*

4. Cyclic Quadrilaterals with Sides in Arithmetic Progression

The analysis in the previous sections began with Heron's formula for the area of a triangle. It is not so well-known that there is a similar formula,

$$A = \sqrt{(s-w)(s-x)(s-y)(s-z)}$$

probably due to Brahmagupta [3], for the area of a cyclic quadrilateral with sides w, x, y, z and semiperimeter s. Let us suppose that the sides are rationals in arithmetic progression, so we may write them $b-d, b, b+d, b+2d$. We shall assume $d \neq 0$ to avoid the trivial

case of the quadrilateral being a square; and thus $0 < d < b$. Then $s = 2b + d$ and the area becomes

$$A = \sqrt{(s+2d)(s+d)s(s-d)} \quad .$$

We wish to determine d and s so that A is rational; thus we seek rational points satisfying the equation

$$A^2 = s^4 + 2s^3d - s^2d^2 - 2sd^3.$$

Dividing through by $(d/2)^4$ and setting $W = 4A/d^2$ and $S = 2s/d$ gives

$$W^2 = S^4 + 4S^3 - 4S^2 - 16S \ .$$

Now translate to remove the cubic term and then use Mordell's birational transformation [6] to convert to a cubic in Weierstrass form. After a further rational transformation, we arrive at the elliptic curve

$$E: \quad y^2 = x(x-1)(x+3)$$

and to solve our problem we must find its group $E(\mathbf{Q})$ of rational points.

We first find the torsion subgroup of $E(\mathbf{Q})$. There are 8 points easily discovered by inspection: $(0,0), (1,0), (-3,0)$ (the obvious points of order 2), $\mathcal{O}$ and $(3, \pm 6), (-1, \pm 2)$ which a calculation shows to be points of order 4. Since the discriminant $\Delta = 2^8 3^2$ we have good reduction modulo 5; we find that $\left|E(\mathbf{F}_5)\right| = 8$ and so there are no more points of finite order. Thus $E_{tors}(\mathbf{Q}) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z}$.

To find the rank of $E(\mathbf{Q})$ we search for solutions to the homogeneous spaces of $E(\mathbf{Q})$ and its 2-isogenous curve

$$\overline{E}(\mathbf{Q}): \quad Y^2 = X(X^2 - 4X + 16).$$

As usual, α will denote the 2-descent homomorphism from E to $\mathbf{Q}^*/\mathbf{Q}^{*2}$ and $\overline{\alpha}$ the corresponding mapping for $\overline{E}$. Then the rank is given by the formula [7]

$$2^{r(E)} = \frac{\left|\alpha(E(\mathbf{Q}))\right|\left|\overline{\alpha}(\overline{E}(\mathbf{Q}))\right|}{4}.$$

Let C_d and $\overline{C}_D$ denote the homogeneous spaces corresponding to $E(\mathbf{Q})$ and $\overline{E}(\mathbf{Q})$ respectively. Then the value of $\left|\alpha(E(\mathbf{Q}))\right|$ is the number of squarefree d dividing 3 such that

$$C_d: \quad dt^2 = d^2r^4 + 2dr^2s^2 - 3s^4$$

has at least one non-trivial solution over $\mathbf{Q}$. Similarly, $\left|\overline{\alpha}(\overline{E}(\mathbf{Q}))\right|$ is the number of squarefree D dividing 16 for which

$$\overline{C}_D: \quad DT^2 = D^2R^4 - 4DR^2S^2 + 16S^4$$

has a non-trivial solution. Solutions are easily found for $d = +1, -1, +3, -3$ and for $D = 1$ while straightforward calculation shows that there are no solutions for any other values of D. Thus $|\alpha(E(\mathbf{Q}))| = 4$ and $|\overline{\alpha}(\overline{E}(\mathbf{Q}))| = 1$ and the formula then shows that the rank of E is 0. Thus there are no further rational solutions.

As a result, we only need to check whether the values of b and d corresponding to the torsion points actually provide examples of Heron quadrilaterals. Retracing our steps through the various substitutions yields the following results:

$(3,6)$	$d = b$
$(3,-6)$	$d = -2b$
$(-1,2)$	$d = -b$
$(-1,2)$	$b = 0$
$(-3,0)$	$d = -2b$
$(1,0)$	$b = \infty$
$(0,0)$	$d = -2b$

Since none of these satifies the condition $0 < d < b < \infty$, we have proved the following

THEOREM 3. *There are no non-trivial cyclic quadrilaterals with rational area and having rational sides forming a non-trivial arithmetic progression.*

We note that there are 3 essentially different possible configurations for a quadrilateral with 4 distinct sides, but, as is obvious from Heron's formula, the area is independent of the configuration.

5. Cyclic Quadrilaterals with Sides in Geometric Progression

In this section we deal with cyclic quadrilaterals of rational area and having sides whose lengths are in geometric progression. Proceeding as above, we find the area expressed as

$$A = \frac{a^2}{4}\sqrt{\Delta}$$

so that we must find values of r for which Δ is a rational square, where

$$\Delta = (-1 + r + r^2 + r^3)(1 - r + r^2 + r^3)(1 + r - r^2 + r^3)(1 + r + r^2 - r^3).$$

Setting $r = m/n$, where m and n are relatively prime, and multiplying by n^{12} yields the diophantine equation

$$y^2 = (m^3 + m^2n + mn^2 - n^3)(m^3 + m^2n - mn^2 + n^3)(m^3 - m^2n + mn^2 + n^3)$$
$$(-m^3 + m^2n + mn^2 + n^3).$$

Now a mod 4 argument shows that m and n are both odd, so that each of the 4 terms above is even. It is easy to see that no two of these terms can have a common prime divisor other than 2. And since each term is congruent to 2 (mod 4), the GCD of any pair of terms is precisely 2. Since the product is a square, each term must be twice a square. Therefore, the product of any pair of terms is a square. So we have, for example,

$$q^2 = (m^3 + m^2n + mn^2 - n^3)(m^3 - m^2n + mn^2 + n^3).$$

Dehomogenising this equation by dividing by n^6 and setting $u = q/n^3$ gives

$$u^2 = r^6 + r^4 + 3r^2 - 1$$

which becomes an elliptic curve upon setting $v = r^2$:

$$u^2 = v^3 + v^2 + 3v - 1.$$

Translating by $v = v_1 - 1/3$ to remove the square term, and then substituting $s = 27u$ and $t = 9v_1$ finally gives the curve in standard form:

$$E: \quad s^2 = t^3 + 216t - 1404.$$

Now we wish to find the group of rational points on this curve. We note that $P = (12, 54)$ is on the curve and that P has order 3. This point corresponds to $r = 1$ and so gives the trivial case of the GP which corresponds to the quadrilateral being a square. The discriminant of E is $-2^4 3^9 41$. Thus there is good reduction mod any prime $p \neq 2, 3, 41$. Reducing mod 5 yields $|E(\mathbf{F}_5)| = 9$ and reducing mod 7 yields $|E(\mathbf{F}_7)| = 6$, so that $|E_{tor}(\mathbf{Q})| = 3$. A calculation using APECS shows the rank of this curve to be 0 and so $\pm P$ are the only rational points on E. Thus we have the following

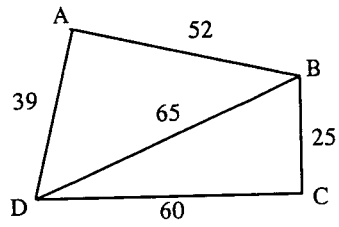
THEOREM 4. *There are no non-trivial cyclic quadrilaterals with rational area having sides in geometric progression.*

6. SOME EXAMPLES

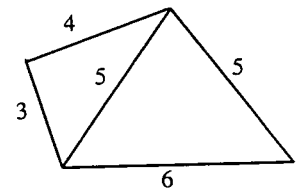
Theorems 3 and 4 depend on the fact that the quadrilateral is cyclic. Some such hypothesis is necessary to guarantee the non-existence of Heron quadrilaterals. For example, the quadrilateral Q2 in Figure 1 has rational sides in arithmetic progression and has a rational area of 18. (It is constructed by fitting together three 3-4-5 right-angled triangles.) Two other similar examples are given by quadrilaterals with sides (51,74,97,120) and (241,409,577,745).

On the other hand, it is easy to construct cyclic quadrilaterals with rational area (whose sides are not in A.P. or G.P.). Quadrilateral Q1 in Figure 1 has right angles at A and C and is one of infinitely many such examples. A preliminary search has not found

any examples of cyclic quadrilaterals with sides in geometric progression, but we have little evidence yet on which to base a conjecture as to whether or not they exist.



Q1



Q2

Figure 1

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