Stability Analysis of Networked Control Systems Subjected to Packet-dropouts and Finite Level Quantization

Yumiko Ishido [*] **Daniel E. Quevedo** [**] **Kiyotsugu Takaba** [*]

[*] *Department of Applied Mathematics and Physics,*
Kyoto University, Kyoto 606-8501, Japan,
(e-mail: {ishido, takaba}@amp.i.kyoto-u.ac.jp)
[**] *School of Electrical Engineering & Computer Science,*
The University of Newcastle, Callaghan, NSW 2308, Australia
(e-mail: dquevedo@ieee.org)

Abstract: In this paper, we study the stability of a networked control system involving signal quantization with finitely many levels and a bounded number of consecutive packet-dropouts. To compensate for the effect of packet-dropouts, the controller-encoder sends a packet which contains possible quantized control inputs for finite future steps. At the receiving end, i.e., at the plant actuator side, a buffer decides the actuator input based on the received data. The buffer has memory which is overwritten whenever it receives a packet from the controller. Within this setting, we derive a sufficient condition on quantization parameters for achieving small ℓ^∞ signal ℓ^∞ stability of the feedback system. The stability condition is characterized in terms of the number of quantization levels of the quantizer.

Keywords: Networked control systems, quantized signals, packet-dropouts.

1. INTRODUCTION

In recent years, networked control systems (NCSs) over practical communication channels have been widely studied to clarify relationships between control performance and channel characteristics such as data rate constraints, packet dropouts and delays (see e.g. Nair *et al.* (2007), Hespanha *et al.* (2007) and the references therein.).

Most of the previous works focus either on data-rate constraints (Nair *et al.* (2007), Tatikonda and Mitter (2004) etc.), packet-dropouts (Hespanha *et al.* (2007), Ishii (2008), Hu and Yan (2007) etc.) or delay (Hetel *et al.* (2006), Pan *et al.* (2006) etc.), separately. However, it is more realistic to assume that the NCS should be affected by all of those channel characteristics.

More recently, Tsumura *et al.* (2009), Niu *et al.* (2009) have studied the mean-square stability of NCSs which involve packet-dropouts as well as quantization with a infinite number of levels. These authors employ the logarithmic quantizer due to Elia and Mitter (2001) to study the trade-off between the *coarseness* of the quantizer and the stability of the NCS.

In the present paper, we will take an alternative view of NCSs affected by packet-dropouts and quantization with finitely many levels in a deterministic setting. Note that the NCS under consideration is nonlinear and time-varying because of quantization and packet-dropouts. Hence, we will study the stability of the NCS based on the notion of small ℓ^∞ signal ℓ^∞ stability, which is recently proposed by Ishido and Takaba (2010a). Within this framework, we will derive a sufficient condition, stated in terms of the number of quantization levels of the quantizer, which guarantees stability of the closed loop system, when affected by bounded packet-dropouts.

Notations: The ∞-norm of a vector $x \in \mathbb{R}^n$ and the ℓ^∞ norm of a signal $f : \mathbb{Z}_+ \to \mathbb{R}^n$ are denoted by $\|x\|_\infty$ and $\|f\|_{\ell^\infty}$, respectively. The extended ℓ^∞ space is defined by

$$\ell_e^\infty = \{f : \mathbb{Z}_+ \to \mathbb{R}^m \mid f_\tau \in \ell^\infty, \, \forall \tau \in \mathbb{Z}_+\},$$

where f_τ denotes the truncation of f at time τ:

$$f_\tau(t) = \begin{cases} f(t) & (0 \le t \le \tau), \\ 0 & (\tau < t). \end{cases}$$

If a map $M : \ell_e^\infty \to \ell_e^\infty$ is finite gain ℓ^∞ stable, then its ℓ^∞-gain is denoted with $\|M\|_{\ell^\infty\text{-ind}}$. The ∞-induced norm of a constant matrix A is defined by $\|A\|_{\infty\text{-ind}} := \sup_{x \neq 0} \|Ax\|_\infty / \|x\|_\infty$.

2. SYSTEM DESCRIPTION

We consider the feedback control system depicted in Fig. 1, which involves an unreliable channel, quantizations, and buffering mechanism. This NCS is similar to those considered in Quevedo *et al.* (2007) and Quevedo and Nešić (2010), but incorporates the quantizer in an explicit manner. To compensate for the packet-dropouts, the controller-encoder produces potential control inputs for current and finite future time instants. The potential control inputs are quantized and packetized at the controller-encoder, and then the packet $\mu(t)$ is transmitted to the communication channel.

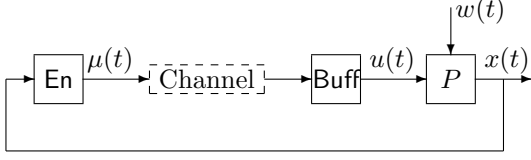The precise description of each component of the NCS is given below.

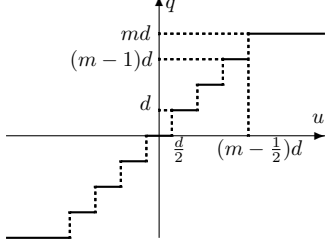Fig. 1. NCS subjected to quantization and packet-dropouts

Fig. 2. Uniform quantizer q

Plant P: The plant P is a discrete-time linear time-invariant (LTI) system whose state-space representation is given by

$$x(t+1) = Ax(t) + Bu(t) + w(t). \quad (1)$$

The signals $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$ and $w(t) \in \mathbb{R}^n$ are the plant state, the actuator input and the process disturbance, respectively. The initial state $x(0)$ is assumed to be zero.

Controller-Encoder En: The controller-encoder sends to the communication channel at time t a control packet $\mu(t) \in \mathbb{R}^N$ which is composed of potential quantized control inputs for the current and $(N-1)$-step future time instants, namely

$$\mu(t) = \begin{bmatrix} q(\hat{u}(t;t)) \\ q(\hat{u}(t+1;t)) \\ \vdots \\ q(\hat{u}(t+N-1;t)) \end{bmatrix}, \quad (2)$$

$$\hat{u}(t+i;t) = K\hat{x}(t+i;t) \quad (i=0,\cdots,N-1) \quad (3)$$

where $\hat{u}(t+i;t) \in \mathbb{R}$ and $\hat{x}(t+i;t) \in \mathbb{R}^n$ are the i-step predictions of the (unquantized) control inputs and the plant states, both of which are produced based on the current plant state $x(t)$. Moreover, $K \in \mathbb{R}^{1 \times N}$ is a constant state-feedback gain and $q: \mathbb{R} \to \mathbb{V} := \{0, \pm d, \pm 2d, \cdots, \pm md\}$ is a static uniform quantizer, where the parameter d represents the step size or fineness of the quantization, and $M := 2m+1$ is the number of the quantization levels.

The quantizer q outputs one of discrete symbols from $\mathbb{V}$, which is the nearest to the real-valued signal $u(t+i;t)$ (see Fig. 2).

$$q(u) = \begin{cases} md & \left(\left(m-\frac{1}{2}\right)d \le u \right) \\ (m-1)d & \left(\left(m-\frac{3}{2}\right)d \le u < \left(m-\frac{1}{2}\right)d \right) \\ \vdots & \vdots \\ 0 & \left(-\frac{1}{2}d \le u < \frac{1}{2}d \right) \\ \vdots & \vdots \\ -md & \left(u < -\left(m-\frac{1}{2}\right)d \right) \end{cases} \quad (4)$$

The predicted states $\hat{x}(t+i;t)$ $(i=1,2,\cdots,N-1)$, are calculated recursively based on the information of $x(t)$,

$\mu(t)$ and the plant dynamics by

$$\begin{cases} \hat{x}(t;t) = x(t), \\ \hat{x}(t+i;t) = A\hat{x}(t+i-1;t) + Bq(\hat{u}(t+i-1;t)) \\ \qquad (i=1,\cdots,N-1). \end{cases} \quad (5)$$

Note that, because of the quantizer q, the packet $\mu(t)$ can take only one of MN different values at each time instant.

Channel: The effect of packet-dropouts over the channel is expressed as the discrete process $\{s(t)\}$ defined by

$$s(t) = \begin{cases} 1 & \text{if packet-dropout does not occur at time } t, \\ 0 & \text{if packet-dropout occurs at time } t. \end{cases}$$

When a packet-dropout does not occur, the channel transmits one of the MN symbols to the buffer without errors.

We denote the time instants when a packet-dropout does not occur, i.e. the transmission is successfully completed, with $\{t_0, t_1, \cdots, t_i, \cdots\}$. That is,

$$s(t) = 1 \Leftrightarrow (\ t = t_i \text{ for some } i \in \mathbb{Z}_+\ ). \quad (6)$$

We also denote with h_i the interval between successful transmissions, namely

$$h_i := t_{i+1} - t_i, \quad i \in \mathbb{Z}_+. \quad (7)$$

Buffer Buff: The buffer Buff decides the actuator input based on the received channel symbols. The state $b(t)$ of the buffer is updated whenever the buffer receives the packet.

$$b(t) = s(t)p_u(t) + (1-s(t))Sb(t-1), \quad b(0) = 0, \quad (8)$$

$$S := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{N \times N}. \quad (9)$$

Then, the buffer decides the actuator input $u(t)$ by

$$u(t) = [1\ 0\ \cdots\ 0]\, b(t). \quad (10)$$

For this feedback control system, we make the following assumptions.

Assumption 1.

$$1 \le h_i \le N \quad \forall i \in \mathbb{Z}_+. \quad (11)$$

This means that the number of consecutive packet-dropouts is bounded by the packet length N.

Assumption 2. The matrix $A_K := A + BK$ is Schur stable.

The idea of employing the buffer to compensate for the effect of packet-dropouts is based on the settings in Quevedo *et al.* (2007) and Quevedo and Nešić (2010). They studied in those papers the Input-to-State Stability (Sontag (1989)) of the feedback system where the plant is controlled over an unreliable channel affected by packet-dropouts. In this paper, we assume the quantization with finitely many levels as well as packet-dropouts at the channel. In this case, it is impossible that the feedback system achieves the finite gain ℓ^p stability because of the quantization (Martins (2006)). To circumvent this difficulty, in the present work, we will study the stability of the NCS based on the notion of *small ℓ^∞ signal ℓ^∞ stability*, which will be described in the following section.

Fig. 3. Feedback system for stability analysis

3. PRELIMINARIES

We briefly revise some basic results on small ℓ^∞ signal ℓ^∞ stability which were introduced in Ishido and Takaba (2010a) and Ishido and Takaba (2010b).

Definition 3. (small ℓ^∞ signal ℓ^∞ stability) A map $H: \ell_e^\infty \to \ell_e^\infty$ is said to be *small ℓ^∞ signal ℓ^∞ stable with level γ and input bound ϵ* if

$$[\![\; \|u_\tau\|_{\ell^\infty} \leq \epsilon \Rightarrow \|(Hu)_\tau\|_{\ell^\infty} \leq \gamma\epsilon \;]\!] \quad \forall u \in \ell_e^\infty, \; \forall \tau \in \mathbb{Z}_+ \tag{12}$$

holds for given positive constants ϵ and γ.

H is simply called small ℓ^∞ signal ℓ^∞ stable if there exist some positive constants ϵ and γ satisfying (12).

The feedback system in Fig. 3 is called small ℓ^∞ signal ℓ^∞ stable if there exist positive constants ϵ and γ such that

$$\left[\!\!\left[\; \left\| \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}_\tau \right\|_{\ell^\infty} \leq \epsilon \Rightarrow \left\| \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}_\tau \right\|_{\ell^\infty} \leq \gamma\epsilon \; \right]\!\!\right]$$
$$\forall r_1, r_2 \in \ell_e^\infty, \; \forall \tau \in \mathbb{Z}_+.$$

In Ishido and Takaba (2010a), the authors recently established the following results for the small ℓ^∞ signal ℓ^∞ stability of the feedback system in Fig. 3.

Proposition 4. For the feedback system in Fig. 3, assume that the following three conditions hold true.

(i) For the sub-system $H_1 : e_1 \mapsto z_1$, there exists a positive constant γ_1 such that

$$\|z_{1\tau}\|_{\ell^\infty} \leq \gamma_1 \|e_{1(\tau-1)}\|_{\ell^\infty} \quad \forall e_1 \in \ell_e^\infty, \; \forall \tau \in \mathbb{Z}_+. \tag{13}$$

(ii) For the sub-system $H_2 : e_2 \mapsto z_2$, there exist positive constants ϵ_2 and γ_2 such that

$$[\![\|e_{2\tau}\|_{\ell^\infty} \leq \epsilon_2 \Rightarrow \|z_{2\tau}\|_{\ell^\infty} \leq \gamma_2\epsilon_2]\!] \quad \forall e_2 \in \ell_e^\infty, \forall \tau \in \mathbb{Z}_+. \tag{14}$$

(iii) $\gamma_1\gamma_2 < 1$

Then, the feedback system is small ℓ^∞ signal ℓ^∞ stable. In particular,

$$[\![\; \|r_\tau\|_{\ell^\infty} \leq \epsilon \Rightarrow (\|z_{1\tau}\|_{\ell^\infty} \leq \delta_1 \text{ and } \|z_{2\tau}\|_{\ell^\infty} \leq \delta_2) \;]\!]$$
$$\forall r \in \ell_e^\infty, \; \forall \tau \in \mathbb{Z}_+$$

holds for

$$\epsilon = \frac{(1 - \gamma_1\gamma_2)\epsilon_1}{1 + \gamma_1}, \quad \delta_1 = \frac{(1 + \gamma_2)\gamma_1\epsilon_2}{1 + \gamma_1}, \quad \delta_2 = \gamma_2\epsilon_2.$$

Proposition 5. Assume that a map $H : \ell_e^\infty \mapsto \ell_e^\infty$ is strictly causal. Then, H satisfies

$$\|u_\tau\|_{\ell^\infty} \leq \gamma \|(Hu)_{\tau-1}\|_{\ell^\infty} \quad \forall u \in \ell_e^\infty, \; \forall \tau \in \mathbb{Z}_+$$

for a given $\gamma > 0$ if and only if it is finite gain ℓ^∞ stable with gain γ, namely

$$\|u_\tau\|_{\ell^\infty} \leq \gamma \|(Hu)_\tau\|_{\ell^\infty} \quad \forall u \in \ell_e^\infty, \; \forall \tau \in \mathbb{Z}_+.$$

4. STABILITY ANALYSIS

The main purpose of this paper is to study the small ℓ^∞ signal ℓ^∞ stability of the closed-loop map from w to x in the NCS of Fig. 1. More specifically, we are interested in:

> Under Assumptions 1 and 2, find a condition on the parameters M, d and N for the existence of positive constants ϵ and γ such that
>
> $$[\![\; \|w_\tau\|_{\ell^\infty} \leq \epsilon \Rightarrow \|x_\tau\|_{\ell^\infty} \leq \gamma\epsilon \;]\!] \quad \forall w \in \ell_e^\infty, \; \forall \tau \in \mathbb{Z}_+ \tag{15}$$
>
> holds for any packet dropout sequence (equivalently, $\{h_i : i \in \mathbb{Z}_+\}$).

Due to packet-dropouts and quantization at the communication channel, the feedback system in Fig. 1 is time-varying (switching due to packet dropouts) and nonlinear.

To tackle this stability problem for the nonlinear switching system, we will derive an alternative feedback representation of the NCS with a new time axis by performing the discrete-time lifting of underlying signals (e.g. Chen and Francis (1995) and the references therein), and by extracting a nonlinearity associated with the quantization. The resulting representation is the feedback interconnection of a strictly causal linear switching system and a nonlinear map which has a bounded level of small ℓ^∞ signal ℓ^∞ stability (Fig. 5). Then, by applying Proposition 4 to this feedback interconnection, we will derive a sufficient condition on M, d and N for achieving (15) for any possible packet-dropout sequence.

4.1 Discrete-time lifting and extraction of quantization nonlinearity

From (2), (6), (8), (9) and (10), the actuator inputs over the interval $[t_i, t_{i+1})$ can be simply written as

$$u(t_i + j) = q(\hat{u}(t_i + j; t_i))$$
$$(i \in \mathbb{Z}_+, \; j = 0, 1, \cdots, h_i - 1). \tag{16}$$

Furthermore, we denote the quantization error with

$$v(t_i + j) := q(\hat{u}(t_i + j; t_i)) - \hat{u}(t_i + j; t_i) \tag{17}$$

for $i \in \mathbb{Z}_+$ and $j = 1, 2, \cdots, h_i - 1 (\leq N - 1)$. Then, it follows from (5) and (3) that

$$\hat{x}(t_i + j; t_i) = A_K^j x(t_i) + \sum_{l=0}^{j-1} A_K^{j-l-1} B v(t_i + l) \tag{18}$$

for $i \in \mathbb{Z}_+$ and $j = 1, 2, \cdots, h_i - 1 (\leq N - 1)$.

Similarly, it can be shown from (1), (3), (16) and (18) that

$$x(t_i + j) = \hat{x}(t_i + j) + \sum_{l=0}^{j-1} A^{j-l-1} w(t_i + l) \tag{19}$$

for $i \in \mathbb{Z}_+$ and $j = 1, 2, \cdots, h_i - 1 (\leq N - 1)$.

By lifting the underlying signals, we obtain

$$\xi[i] := x(t_i) \in \mathbb{R}^n, \tag{20}$$

$$\tilde{w}[i] := \begin{bmatrix} w(t_i) \\ w(t_i + 1) \\ \vdots \\ w(t_i + h_i - 1) \end{bmatrix} \in \mathbb{R}^{nh_i}, \quad \tilde{v}[i] := \begin{bmatrix} v(t_i) \\ v(t_i + 1) \\ \vdots \\ v(t_i + h_i - 1) \end{bmatrix} \in \mathbb{R}^{h_i}, \tag{21}$$
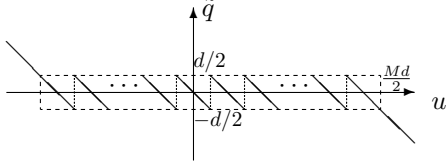
Fig. 4. Quantization error $\tilde{q}$

$$\tilde{u}[i] := \begin{bmatrix} \hat{u}(t_i; t_i) \\ \hat{u}(t_i + 1; t_i) \\ \vdots \\ \hat{u}(t_i + h_i - 1; t_i) \end{bmatrix} \in \mathbb{R}^{h_i}. \tag{22}$$

It is then easily verified from (16)–(22) that the NCS in Fig. 1 is expressed as the linear fractional transformation (LFT) of the linear system G and the nonlinear map Q (see Fig. 5(a)). The linear system G is defined by the state-space equation

$$G : \begin{cases} \xi[i + 1] = A_{h_i} \xi[i] + B_{v,h_i} \tilde{v}[i] + B_{w,h_i} \tilde{w}[i], \ \xi[0] = 0, \\ \tilde{u}[i] = C_{u,h_i} \xi[i] + D_{v,h_i} \tilde{v}[i], \end{cases}$$

where i denotes the new time index, and

$$A_h := A_K^h \in \mathbb{R}^{n \times n}, \quad C_{u,h} := \begin{bmatrix} K \\ KA_K \\ \vdots \\ KA_K^{h-1} \end{bmatrix} \in \mathbb{R}^{h \times n},$$

$$B_{v,h} := \begin{bmatrix} A_K^{h-1}B & \cdots & A_K B & B \end{bmatrix} \in \mathbb{R}^{n \times h},$$

$$B_{w,h} := \begin{bmatrix} A^{h-1} & \cdots & A & I \end{bmatrix} \in \mathbb{R}^{n \times nh},$$

$$D_{v,h} := \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ KB & 0 & \cdots & \cdots & 0 \\ KA_K B & KB & \ddots & & \vdots \\ \vdots & & \ddots & 0 & 0 \\ KA_K^{h-2}B & \cdots & KA_K B & KB & 0 \end{bmatrix} \in \mathbb{R}^{h \times h},$$

$$(h = 1, 2, \ldots, N)$$

On the other hand, Q is the memoryless diagonal map defined by

$$Q : \mathbb{R}^{h_i} \to \mathbb{R}^{h_i}; \ Q(u) = \begin{bmatrix} \tilde{q}(u_1) \\ \vdots \\ \tilde{q}(u_{h_i}) \end{bmatrix} \text{ for } u = \begin{bmatrix} u_1 \\ \vdots \\ u_{h_i} \end{bmatrix}, \quad (23)$$

where

$$\tilde{q}(u) := q(u) - u \tag{24}$$

is the static map associated with the quantization error (see Fig. 4).

It should be noted that both of G and Q are switching systems because the dimensions of the signals and the coefficient matrices change on the occurrence of packet-dropouts which cannot be predicted in advance.

By defining $\eta := \tilde{u} - D_{v,h_i} \tilde{v}$ and

$$\hat{Q} := (I - Q \circ D_{v,h_i})^{-1} \circ Q, \tag{25}$$

we further obtain the LFT representation of Fig. 5(b), where G_{sc} is the linear switching system G_{sc} defined by

$$G_{sc} : \begin{cases} \xi[i + 1] = A_{h_i} \xi[i] + B_{v,h_i} \tilde{v}[i] + B_{w,h_i} \tilde{w}[i], \ \xi[0] = 0, \\ \eta[i] = C_{u,h_i} \xi[i]. \end{cases}$$
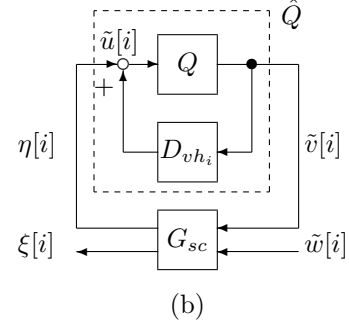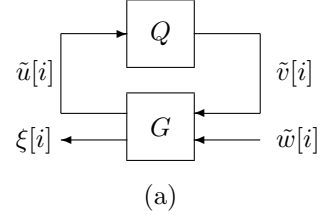$$\tag{26}$$

(a)

(b)

Fig. 5. LFT representations of the NCS via lifting

For ease of later discussion, we define

$$\gamma_D := \max_{h \in [1,N]} \|D_{v,h}\|_{\infty\text{-ind}} = \sum_{j=0}^{N-2} |KA_K^j B|. \tag{27}$$

Lemma 6. If

$$M > \gamma_D, \tag{28}$$

then $\hat{Q} : \eta \mapsto \tilde{v}$ satisfies

$$[\![\|\eta_\tau\|_{\ell^\infty} \le \epsilon_Q \Rightarrow \|\tilde{v}_\tau\|_{\ell^\infty} \le \gamma_Q \epsilon_Q]\!] \ \forall \eta \in \ell_e^\infty, \forall \tau \in \mathbb{Z}_+$$

for

$$\epsilon_Q = (M - \gamma_D)\frac{d}{2}, \qquad \gamma_Q = \frac{1}{M - \gamma_D}, \tag{29}$$

and for any sequence $\{h_i : i \in \mathbb{Z}_+\}$.

Since the lifting operation is ℓ^∞-norm preserving (e.g. $\|w\|_{\ell^\infty} = \|\tilde{w}\|_{\ell^\infty}$, etc), the next lemma shows that the small ℓ^∞ signal ℓ^∞ stabilities of the NCS in in Fig. 1 and of the lifted system in Fig. 5(b) are equivalent.

Lemma 7. The following two statements are equivalent.

(i) The closed-loop map from w to x in Fig. 1 is small ℓ^∞ signal ℓ^∞ stable, namely there exist positive constants ϵ and γ such that (15) is satisfied for any sequence $\{h_i : i \in \mathbb{Z}_+\}$.

(ii) For the feedback system in Fig. 5(b), there exist positive constants ϵ' and γ' such that

$$[\![\|\tilde{w}_\tau\|_{\ell^\infty} \le \epsilon' \Rightarrow \|\xi_\tau\|_{\ell^\infty} \le \gamma' \epsilon']\!] \ \forall \tilde{w} \in \ell_e^\infty, \ \forall \tau \in \mathbb{Z}_+$$

holds for any sequence $\{h_i : i \in \mathbb{N}\}$.

Next, by introducing auxiliary inputs $r_{11}, r_{12}, r_{22}, z_{22}$ as in Fig. 6(a), the stability analysis of the NCS in Fig. 5(b) is reduced to that of the feedback interconnection of G_{sc} and Δ. To be more specific, we define

$$r_1[i] = \begin{bmatrix} r_{11}[i] \\ \tilde{w}[i] \end{bmatrix} \in \mathbb{R}^{h_i + nh_i}, \quad r_2[i] = \begin{bmatrix} r_{12}[i] \\ r_{22}[i] \end{bmatrix} \in \mathbb{R}^{h_i + n},$$

$$z_1[i] = \begin{bmatrix} \eta[i] \\ \xi[i] \end{bmatrix} \in \mathbb{R}^{h_i + n}, \quad z_2[i] = \begin{bmatrix} \tilde{v}[i] \\ z_{22}[i] \end{bmatrix} \in \mathbb{R}^{h_i + nh_i},$$

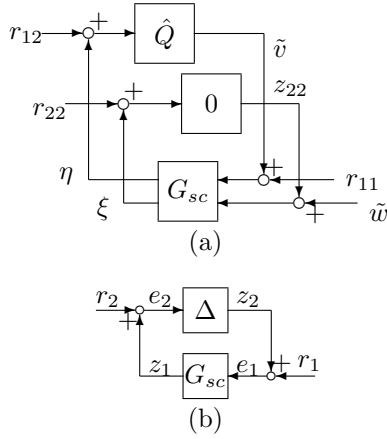$$e_1[i] = r_1[i] + z_2[i], \quad e_2[i] = r_2[i] + z_1[i].$$

Fig. 6. Feedback representations with auxiliary inputs

Fig. 6(a) is further rewritten as in Fig. 6(b), where

$$G_{sc} : \begin{cases} \xi[i+1] = A_h\xi[i] + B_h e_1[i] \\ z_1[i] = C_h\xi[i] \end{cases} \tag{30}$$

$$B_h := [B_{v,h} \ B_{w,h}] \in \mathbb{R}^{n \times (n+nh)},$$

$$C_h := \begin{bmatrix} C_{u,h} \\ I \end{bmatrix} \in \mathbb{R}^{(h+n) \times n},$$

and Δ is the nonlinear diagonal map defined by

$$\Delta : \mathbb{R}^{h_i+nh_i} \to \mathbb{R}^{h_i+nh_i}; \ \Delta(e_2) = \begin{bmatrix} \hat{Q}(e_{12}) \\ 0 \end{bmatrix} \text{ for } e_2 = \begin{bmatrix} e_{12} \\ e_{22} \end{bmatrix}. \tag{31}$$

Then, the stability of the NCS in Fig. 5(b) is related to that of the feedback system in Fig. 6(b) by the following lemma.

Lemma 8. Assume that the feedback system in Fig. 6(b) is small ℓ^∞ signal ℓ^∞ stable. Then, the closed-loop map from $\tilde{w}$ to ξ in Fig. 5(b) is also small ℓ^∞ signal ℓ^∞ stable, namely, there exist positive constants ϵ and γ such that

$$[\![\ \|\tilde{w}_\tau\|_{\ell^\infty} \le \epsilon \Rightarrow \|\xi_\tau\|_{\ell^\infty} \le \gamma\epsilon \ ]\!] \ \forall \tilde{w} \in \ell_e^\infty, \ \forall \tau \in \mathbb{Z}_+ \tag{32}$$

holds for any sequence of $\{h_i : i \in \mathbb{Z}_+\}$.

Proof. By assumption, there exist positive constant ϵ and γ such that, for the feedback system of Fig. 6(b),

$$\left[\!\!\left[\ \left\| \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}_\tau \right\|_{\ell^\infty} \le \epsilon \Rightarrow \left\| \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}_\tau \right\|_{\ell^\infty} \le \gamma\epsilon \ \right]\!\!\right]$$
$$\forall r_1, r_2 \in \ell_e^\infty, \ \forall \tau \in \mathbb{Z}_+$$

holds for any sequence of $\{h_i : i \in \mathbb{Z}_+\}$. It is easily seen from the definitions of r_1, r_2, z_1, z_2 and Δ that taking $r_{11} = 0$, $r_{21} = 0$ and $r_{12} = 0$ in the above equation yields (32) for the closed-loop map of Fig. 5(b). This completes the proof. $\qquad \square$

We now derive a sufficient condition of M and d for the small ℓ^∞ signal ℓ^∞ stability of the system in Fig. 6(b).

Lemma 9. Assume that the linear switching system $G_{sc} : z_1 \mapsto e_1$ is finite gain ℓ^∞ stable with gain γ_G, namely,

$$\|z_{1\tau}\|_{\ell^\infty} \le \gamma_G\|e_{1\tau}\|_{\ell^\infty} \quad \forall e_1 \in \ell_e^\infty, \ \forall \tau \in \mathbb{Z}_+ \tag{33}$$

holds for any sequence of $\{h_i : i \in \mathbb{Z}_+\}$. Then, the feedback system in Fig. 6(b) is small ℓ^∞ signal ℓ^∞ stable if

$$M > \gamma_G + \gamma_D. \tag{34}$$

In particular,

$$\left[\!\!\left[\ \left\| \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}_\tau \right\|_{\ell^\infty} \le \epsilon \Rightarrow (\|z_{1\tau}\|_{\ell^\infty} \le \delta_1 \text{ and } \|z_{2\tau}\|_{\ell^\infty} \le \delta_2) \ \right]\!\!\right]$$
$$\forall r_1, r_2 \in \ell_e^\infty, \ \forall \tau \in \mathbb{Z}_+ \tag{35}$$

holds for

$$\epsilon = \frac{M - \gamma_G - \gamma_D}{(1 + \gamma_{G_{sc}})(M - \gamma_D)}, \tag{36}$$

$$\delta_1 = \frac{d\gamma_G(M - \gamma_D + 1)}{2(1 + \gamma_G)}, \quad \delta_2 = \frac{d}{2}. \tag{37}$$

Proof. Since $G_{sc} : e_1 \mapsto z_1$ is strictly causal and has finite gain γ_G, we see from Proposition 5 that

$$\|z_{1\tau}\|_{\ell^\infty} \le \gamma_G\|e_{1(\tau-1)}\|_{\ell^\infty} \quad \forall e_1 \in \ell_e^\infty \ \forall \tau \in \mathbb{Z}_+. \tag{38}$$

Since γ_G and γ_D are positive, (34) implies $M - \gamma_D > 0$. It thus follows from (31) and Lemma 6 that $\Delta : e_2 \mapsto z_2$ satisfies

$$[\![\ \|e_{2\tau}\|_{\ell^\infty} \le \epsilon_Q \Rightarrow \|z_{2\tau}\|_{\ell^\infty} \le \gamma_Q\epsilon_Q \ ]\!]$$
$$\forall e_2 \in \ell_e^\infty, \ \forall \tau \in \mathbb{Z}_+ \tag{39}$$

for any sequence $\{h_i : i \in \mathbb{Z}_+\}$, where ϵ_Q and γ_Q are defined in (29). Moreover,

$$\gamma_G\gamma_Q < 1 \tag{40}$$

immediately follows from (29) and (34). By Proposition 4, (38), (39) and (40) prove the small ℓ^∞ signal ℓ^∞ stability of the feedback system in Fig. 6(b). $\qquad \square$

Consequently, we obtain the main result of this paper by combining Lemmas 7, 8 and 9.

Theorem 10. Under Assumptions 1 and 2, assume that

$$M > \gamma_G + \gamma_D \tag{41}$$

is satisfied. Then, there exist positive constants ϵ and γ such that the closed-loop map from w to x in the NCS of Fig. 1 satisfies (15) for any packet dropout sequence.

4.2 Computation of the upper bound on $\|G_{sc}\|_{\ell^\infty\text{-ind}}$

We here present a method for estimating γ_G satisfying (33) based on the reachable set analysis. Although G_{sc} is a switching system, its linearity enables us to compute an upper bound on γ_G by using a reachable set for inputs with a fixed bound. See Shingin and Ohta (2004) for the details of the algorithm for computing reachable sets for LTI systems.

Theorem 11. Suppose that there exist positive constants β_h ($h = 1, 2, \cdots, N$), γ, nonnegative constants $\sigma_{h,j}$ ($h = 1, 2, \cdots, N, j = 1, \cdots, (n+1)h$) and a positive definite matrix X satisfying

$$\begin{bmatrix} A_h^T X A_h - (1 - \beta_h)X & A_h^T X B_h \\ B_h^T X A_h & B_h^T X B_h - \beta_h\Sigma_h \end{bmatrix} \le 0$$
$$(h = 1, 2, \cdots, N), \tag{42}$$

$$0 < \beta_h < 1 \quad (h = 1, 2, \cdots, N), \tag{43}$$

$$\Sigma_h := \begin{bmatrix} \sigma_{h,1} & & 0 \\ & \ddots & \\ 0 & & \sigma_{h,(n+1)h} \end{bmatrix}, \quad \sum_{j=1}^{(n+1)h} \sigma_{h,j} = 1$$
$$(h = 1, 2, \cdots, N), \tag{44}$$

$$X \geq \gamma^{-2}(A_K^{h-1})^T K^T K A_K^{h-1}$$
$$(h = 1, 2, \cdots, N), \qquad (45)$$
$$X \geq -\gamma^{-2} f_k f_k^T \quad (k = 1, 2, \cdots, n) \qquad (46)$$

where f_k denotes the k-th standard basis in $\mathbb{R}^n$. Then, the linear switching system G_{sc} is finite gain ℓ^∞ stable, and

$$\gamma_G \leq \gamma.$$

Proof. Omitted for the space limitation.

We here give a remark on the meaning of the matrix inequality condition in Theorem 11. The inequalities (42)–(44) ensure that the ellipsoid

$$\mathcal{E}(X, \epsilon) := \{\xi \in \mathbb{R}^n : \xi^T X \xi \leq \epsilon^2\}$$

is an outer approximation of the reachable set of G_{sc} : $e_1 \mapsto z_1$ for the input e_1 whose magnitude is bounded by ϵ, i.e. $\|e_1[i]\|_\infty \leq \epsilon \; \forall i \in \mathbb{Z}_+$. Moreover, (45) and (46) imply that $\xi[i] \in \mathcal{E}(X, \epsilon) \Rightarrow \|z_1[i]\|_\infty \leq \gamma\epsilon \; (h = 1, \ldots, N)$. Hence, we see that (42)–(46) give an upper bound γ on the ℓ^∞-gain $\gamma_G = \|G_{sc}\|_{\ell^\infty\text{-ind}}$.

Notice that (42)-(46) become Linear Matrix Inequalities (LMIs) in X, γ^{-2} and $\Sigma_1, \ldots, \Sigma_N$ if we fix $\beta_1, \ldots, \beta_N$ to some constants. Thus, we can efficiently find a good upper bound on γ_G by solving the convex programming problem:

$$\hat{\gamma}_G := \inf_{X, \gamma^{-2}, \Sigma_1, \ldots, \Sigma_N} -\gamma^{-2} \text{ subject to (42)–(46)},$$

where $\beta_1, \ldots, \beta_N$ are determined by grid search.

Given an upper bound of $\|G_{sc}\|_{\ell^\infty\text{-ind}}$, we can obtain a sufficient number of quantization levels for achieving the small ℓ^∞ signal ℓ^∞ stability of the NCS, since the stability condition (34) is characterized in terms of the number of quantization levels M. Though the step size d of the quantizer is not involved in (34), it does affect the level and the input bound of small ℓ^∞ signal ℓ^∞ stability of the NCS (see (35)-(37)).

From (27),(28) and Theorem 10, the right hand side of (34) becomes in general large for a large N, which represents a tradeoff between the number of quantization level, or the data rate of the communication channel, and the packet-dropouts for the small ℓ^∞ signal ℓ^∞ stability for the NCS.

5. CONCLUDING REMARKS

We have studied the small ℓ^∞ signal ℓ^∞ stability of the NCS where the plant is controlled over the communication channel affected by packet-dropouts and finite level quantization. A sufficient stability condition has been derived in terms the parameters of the quantizer and the packet length. An LMI-based method to numerically check the stability condition has also been developed.

REFERENCES

T. Chen and B. Francis. *Optimal Sampled-Data Control Systems*, Springer, 1995

G.N. Nair, F. Fagnani, S. Zampieri and R.J. Evans. Feedback control under data rate constraints: an overview. *Proc. of IEEE*, 95(1): 108-137, 2007

J.P. Hespanha, P. Naghshtabrizi and Y. Xu. A survey of recent results in networked control systems. *Proc. of the IEEE*, 95(1): 138-162, 2007

H. Ishii. H^∞ control with limited communication and message losses. *Syst. Contr. Lett.*, 57: 322-331, 2008

S. Hu and W.-Y. Yan. Stability analysis of networked control systems with respect to packet loss. *Automatica*, 43(7): 1243-1248, 2007

S. Tatikonda and S.K. Mitter. Control under communication constraints. *IEEE Trans. Automat. Contr.*, 49(7): 1056-1068, 2004

L. Hetel, J. Daafouz and C. Iung. Stabilization of arbitrary switched linear systems with unknown time-varying delays. *IEEE Trans. Automat. Contr.*, 51(10): 1668-1674, 2006

Y.-J. Pan, H.J. Marquez and T. Chen. Stabilization of remote control systems with unknown time varying delays by LMI techniques. *Int. Journal of Control*, 79(7): 752-763, 2006

K. Tsumura, H. Ishii and H. Hoshina. Tradeoffs between quantization and packet loss in networked control of linear systems. *Automatica*, 45(12): 2963-2970, 2009

Y. Niu, T. Jia, X. Wang and F. Yang. Output-feedback control design for NCSs subject to quantization and dropout. *Information Sciences*, 179: 3804-3813, 2009

N. Elia and S.K. Mitter. Stabilization of linear systems with limited information. *IEEE Trans. Automat. Contr.*, 45: 1279-1289, 2001

N.C. Martins. Finite gain l_p stabilization requires analog control, *Syst. Contr. Lett.*, 55: 949-954, 2006

E.D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Contr.*, 34(4):435-443, 1989

D.E. Quevedo, E.I. Silva and G.C. Goodwin. Packetized predictive control over erasure channels. *Proc. of American Control Conference*, 2007

D.E. Quevedo and D. Nešić. Input-to-state stability of packetized predictive control over unreliable networks affected by packet-dropouts. *IEEE Trans. Automat. Contr.*, to be published, 2010

Y. Ishido and K. Takaba. Robust ℓ^p stability and robust ℓ^∞ stabilization over a rate-limited digital channel, *SICE Journal of Control, Measurement, and System Integration*, 3(2): 101-110, 2010

Y. Ishido and K. Takaba. A new framework for robust stability analysis of quantized feedback systems, *Proc. of 2010 American Control Conference*, 2010

H. Shingin and Y. Ohta. Optimal invariant sets for discrete-time systems: approximation of reachable sets for bounded inputs, *Proc. of 10th IFAC/IFORS/IMACS/IFIP Symposium on Large Scale Systems: Theory and Applications*, 401-406, 2004