3

Generalized Sampled-data Hold Functions

Generalized Sampled-data Hold Functions [e.g., Kabamba, 1987, Bai and Dasgupta, 1990, Yan et al., 1994, Er et al., 1994] have been proposed as an approach to several control problems that do not have answers with analog LTI, or traditional sampled-data settings based on the ZOH. GSHF-based control schemes are sampled-data systems where the D-A conversion is performed using a special waveform instead of the constant function generated by the ZOH (see Figure 3.1). The choice of this waveform is an additional degree of freedom incorporated to the design, and it seems to give a number of advantages over other control schemes. For example, it has been recently shown that if there exist a solution to the H_∞ control problem for sampled-data systems, then this solution can be implemented by a GSHF following a LTI discrete controller. [e.g., Sun et al., 1993].

However, serious robustness and sensitivity problems associated with the use of GSHFs have been pointed out by some authors Feuer and Goodwin [1994], Zhang and Zhang [1994] showing that many of the most impressive features of GSHFs come along with quite undesirable "side-effects". For example, Feuer and Goodwin [1994] have shown that the arbitrary shaping of the sampled frequency response of a system by means of a GSHF necessarily relies on the generation of high frequency components in the continuous-time output. This exposes the mechanism by which sensitivity and robustness properties of the system are compromised, since in practice high frequency uncertainty is very common.

Furthermore, as we shall see in Chapter 4, there are essential continuous-time design limitations that are inherited by the sampled-data system, irrespective of the particular discretization method employed. Particularly linked to these issues are the frequency response and the zeros of the hold device. It turns out, for example, that "non-minimum phase" holds, i.e., holds with zeros in $\mathbb{C}^+$, impose extra limitations in the achievable continuous-time performance of the system. These "non-minimum phase" zeros of the hold may also lead to poorly conditioned discretized systems, as has been discussed by Middleton and Freudenberg [1995] and Middleton and Xie [1995].

In this chapter, we study the frequency response and zero distribution of GSHFs. The results obtained here allow us to go deeper into the understanding

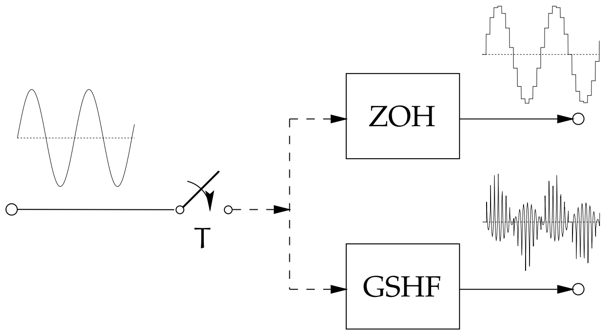

Figure 3.1: D-A conversion with ZOH and GSHF

of the design tradeoffs associated with the use of these devices. For example, one of the key results of this chapter is that holds with "asymmetric" pulse response will necessarily have zeros off the $j\omega$-axis, which may lead to the aforementioned difficulties.

The organization of the chapter is as follows. In §3.1, we collect several properties of the frequency response of a GSHF. Among these properties are some interesting relations between the frequency response of a generalized hold and that of a ZOH. The distribution of zeros of GSHFs is the theme of §3.2. In §3.3, we establish some connections between these zeros and the values of the frequency response on the $j\omega$-axis. Finally, we provide some concluding remarks in §3.4.

3.1 Frequency Response of Generalized Sampled-data Holds

The most standard and simplest D-A converter in digital control implementations is the ZOH. Given a discrete input sequence $\{u_k\}_{k=0}^{\infty}$, the ZOH is defined by

$$u(t) = u_k, \quad \text{for } kT \leq t < (k+1)T.$$

In particular, the ZOH can be seen as a particular case of the GSHF defined in (2.3) with the hold function

$$h(t) = \begin{cases} 1 & t \in [0, T) \\ 0 & \text{otherwise} \end{cases}$$

(see Figure 3.2).

The idea of a GSHF is to allow h to be some suitably chosen function instead of just holding the discrete values constant during the sampling interval. In this way a new degree of freedom is introduced in the sampled-data control design problem, in addition to the choice of the discrete controller.

In this section we present some preliminary results concerning the frequency response of a GSHF. In Subsections 3.1.1 and 3.1.2, we obtain some general properties of the frequency response of a GSHF, norms and reconstruction from boundary values. These properties are intimately linked to the fact that GSHF frequency

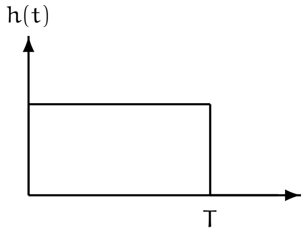

Figure 3.2: Pulse response of a ZOH.

responses are entire functions, as we noted in Subsection 2.1.2. In Subsection 3.1.3 we introduce two important characterizations of GSHFs common in the literature, namely, GSHFs where h is the truncated pulse response of a dynamic system, and GSHFs where h is a piecewise-constant function.

3.1.1 Norms and the Frequency Response of a GSHF

Let H be the frequency response of a GSHF defined by (2.4). Since h is of BV on $[0, T)$ by Assumption 1, then h is in $L_2(0, T)$, that is

$$\|h\|_2^2 = \int_0^T |h(t)|^2 \, dt < \infty.$$

A well-known result of Paley and Wiener [1934] says that the Laplace transform of a square integrable function vanishing outside the interval $(-T, T)$ is an entire function of order 1 and type[1] T. Moreover, since h actually vanishes outside $[0, T)$, the function H is uniformly bounded in the closed right half plane. Indeed, we can easily check this from the following inequalities, which follow as a trivial application of Cauchy-Schwarz Inequality.

$$\begin{aligned} |H(re^{j\theta})|^2 &\leq \left(\int_0^T |h(t)|^2 \, dt\right)\left(\int_0^T e^{-2rt\cos\theta} \, dt\right) \\ &= \|h\|_2^2 \left(\frac{1 - e^{-2rT\cos\theta}}{2r\cos\theta}\right). \end{aligned} \tag{3.1}$$

Now, we can see from (3.1) that if the angle θ is on the interval $[-\pi/2, \pi/2]$, then

$$|H(re^{j\theta})| \leq \sqrt{T}\,\|h\|_2. \tag{3.2}$$

In particular, (3.2) tells us that if the infinity norm, $\|H\|_\infty = \sup_\omega |H(j\omega)|$, is large, then the 2-norm,

$$\|H\|_2 = \left(\int_{-\infty}^{\infty} |H(j\omega)|^2 \, d\omega\right)^{1/2},$$

[1]The order and type of an entire function quantify its growth properties, see Appendix B for a brief description.

will also be large, since by Parseval's Formula $\|H\|_2 = \sqrt{2\pi}\,\|h\|_2$.

Another interesting connection between frequency and time domain values is given by the following lemma [*cf.* Yamamoto and Araki, 1994, Lemma 3.3].

Lemma 3.1.1 (Parseval's Equality for Holds)
For any real number ω *and any* H *defined by* (2.4)*,*

$$\frac{1}{T}\sum_{k=-\infty}^{\infty} |H(j\omega + jk\omega_s)|^2 = \|h\|_2^2 \tag{3.3}$$

Proof: Consider the function $f_\omega(t) = h(t)\, e^{-j\omega t}$, with support on the interval $[0, T)$. Its Fourier series representation is

$$f_\omega(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{jk\omega_s t}, \qquad \text{for } t \text{ in } [0, T)$$

where the Fourier coefficients are

$$\begin{aligned} c_k &= \frac{1}{T}\int_0^T f_\omega(t)\, e^{-jk\omega_s t}\, dt \\ &= \frac{1}{T} H(j\omega + jk\omega_s) \end{aligned} \tag{3.4}$$

Hence, by Parseval's Formula we have that

$$\frac{1}{T}\int_0^T |f_\omega(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

The result is then obtained by noting that $|f_\omega(t)| = |h(t)|$, and replacing c_k from (3.4). □

An interpretation in terms of frequency aliasing can be given to the above result. Suppose that $H(0) \neq 0$, i.e., the hold has non-zero *DC-gain*, and (without loss of generality) assume that $H(0) = 1$. If $|H(j\omega)|$ has a large peak, say $\|H\|_\infty \gg 1$, then from (3.2) and (3.3) follows that

$$\sum_{k=-\infty}^{\infty} |H(j\omega + jk\omega_s)|^2 \gg 1. \tag{3.5}$$

Hence, evaluation of (3.5) at small values of ω still gives a large sum, and so there must be a significant number of other terms ($k \neq 0$) adding to $|H(j\omega)|$ to give a large 2-norm. Thus, a peak of $|H(j\omega)|$ necessarily implies a lot of frequency "folding" going on. In particular, note that since H_{ZOH} has zeros at integer multiples of the sampling frequency ω_s, then the ZOH has the minimum L_2-norm over all the holds that satisfy $H(0) = 1$.

Yet a last property of GSHFs gives us the "gain" of the hold viewed as an input-output operator. Let $\mathcal{H}$ denote the hold operator mapping ℓ_p to L_p, $1 \leq p \leq \infty$, defined by (2.3). The lemma below is a generalization to GSHFs of a result for the ZOH in Francis [1991].

Lemma 3.1.2 (Input-output norm of a hold operator)
The hold operator $\mathcal{H} : \ell_p \to L_p$ *is bounded and of norm* $\|h\|_p$.

Proof: We prove this for $p < \infty$; the case $p = \infty$ follows similar steps. Let u be a function in L_p, and $v = \{v_k\}_{k=-\infty}^{\infty}$ a sequence in ℓ_p, such that $u = \mathcal{H}v$. Then,

$$\begin{aligned}
\|u\|_p^p &= \int_{-\infty}^{\infty} |u(t)|^p \, dt \\
&= \sum_{k=-\infty}^{\infty} \int_{kT}^{(k+1)T} |h(t-kT)v_k|^p \, dt \\
&= \left(\int_0^T |h(t)|^p \, dt \right) \left(\sum_{k=-\infty}^{\infty} |v_k|^p \right) \\
&= \|h\|_p^p \, \|v\|_p^p .
\end{aligned}$$

□

In particular, Lemma 3.1.2 tells us that the induced norm of the hold operator in the case of bounded-input, bounded-output (BIBO) spaces ($p = \infty$) is precisely $\|h\|_\infty$. Therefore, we see that a large value of $\|h\|_\infty$ implies a "high gain" hold, viewed as a BIBO device. Combining (3.2) with the fact that $\|h\|_2 \le \sqrt{T}\,\|h\|_\infty$, we obtain

$$\|H\|_\infty \le T\,\|h\|_\infty .$$

So, we see that, for a given sampling rate, a large peak in $|H(j\omega)|$ also implies a large BIBO gain. Since the output of the hold is typically the input to the plant, such large gain may introduce serious difficulties due to actuator saturations, present in most real systems Gilbert [1992].

3.1.2 GSHF Frequency Responses from Boundary Values

Analytic functions can be reconstructed from their boundary values by means of integral formulas like Poisson's or Cauchy's [e.g., Hoffman, 1962]. Not surprisingly, since they are entire functions, GSHF frequency responses can be recovered from similar relations. The interesting fact is that the frequency response of a ZOH is involved in these reconstructions. In this subsection we present two results on reconstruction from boundary values of the frequency response of a GSHF.

Denote by H_{ZOH} the response of a ZOH,

$$H_{ZOH}(s) = \frac{1 - e^{-sT}}{s}.$$

The following lemma is a straightforward consequence of the Fourier representation of h [See also Feuer and Goodwin, 1996].

Lemma 3.1.3 (Hold Response from Boundary Values: "Discrete" Version)
For any complex number s *and any* H *defined by* (2.4),

$$H(s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H(jk\omega_s)\, H_{ZOH}(s - jk\omega_s) \tag{3.6}$$

Proof: Expand h into Fourier series,

$$h(t) = \sum_{K=-\infty}^{\infty} c_k\, e^{jk\omega_s t}, \quad \text{where } c_k = \frac{1}{T}\int_0^T h(t)\, e^{-jk\omega_s t}\, dt = H(jk\omega_s). \tag{3.7}$$

Then, the Laplace transform of (3.7) gives

$$H(s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H(jk\omega_s)\, \frac{1 - e^{-sT}}{s - jk\omega_s},$$

completing the proof. □

Interestingly, there exists a — not so obvious — "continuous" version of the above formula, arising from properties of Paley-Wiener spaces of entire functions De Branges [1968].

Lemma 3.1.4 (Hold Response from Boundary Values: "Continuous" Version)
For any complex number s *and any* H *defined by* (2.4),

$$H(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega)\, H_{ZOH}(s - j\omega) d\omega \tag{3.8}$$

Proof: If f is a function that vanishes outside the interval $[-T/2, T/2]$, and its Laplace transform, F, is such that $\int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega < \infty$, then F is an entire function of type $T/2$ [De Branges, 1968, p. 45]. Moreover, for any complex number s,

$$F(s) = \int_{-\infty}^{\infty} F(j\omega)\, \frac{\sinh(j\omega T/2 - sT/2)}{\pi(j\omega - s)}\, d\omega. \tag{3.9}$$

Applying (3.9) to the function $H(s)\, e^{-sT/2}$ gives the result. □

3.1.3 Two Simple Classes of GSHFs

To further study properties of the frequency responses of GSHFs we need to describe them in greater detail. In this subsection we present two different classes of GSHFs that are important for their simple mathematical description. These holds have been suggested by different authors, and were studied in the present formulation by Middleton and Freudenberg [1995].

The first class of GSHFs is characterized by a pulse response h generated as the response of a finite dimensional linear time-invariant system truncated to have support on the interval $[0, T)$ (see Figure 3.3). This family covers, for example, the type of GSHFs suggested by Kabamba [1987] to achieve simultaneous

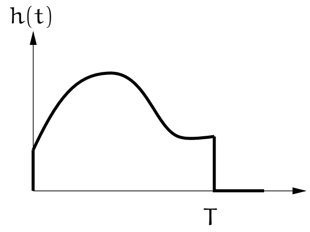

Figure 3.3: Pulse response of a FDLTI GSHF.

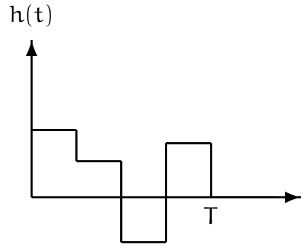

Figure 3.4: Pulse response of a PC GSHF.

stabilization of a finite number of continuous-time plants, decoupling, discrete model matching, discrete simultaneous optimal noise rejection, and arbitrary gain-margin improvement [See also Bai and Dasgupta, 1990, Liu et al., 1992, Haddad et al., 1994, Yang and Kabamba, 1994, Paraskevopoulos and Arvanitis, 1994].

Definition 3.1.1 (Finite Dimensional Linear Time-invariant GSHF)
Given suitably dimensioned matrices K, L and M, we define a finite dimensional linear time-invariant GSHF (FDLTI GSHF) by the pulse response

$$h(t) = Ke^{L(T-t)}M, \qquad \text{for } t \in [0, T). \tag{3.10}$$

◇

FDLTI GSHFs have a simple and convenient model for analysis and design of GSHF-based control systems. Yet, this model still seems an impractical scheme for implementation.

The second class of GSHFs is characterized by a piecewise-constant pulse response function h, typically with a regular partition of N subintervals of the sampling interval $[0, T)$ (see Figure 3.4). Clearly [*e.g. Yan et al., 1994]*, this type of holds can arbitrarily approximate any GSHF of the form (3.10) by taking N sufficiently large and, in addition, appears as a much more feasible alternative for a practical implementation. Holds of this class have been suggested for discrete loop transfer recovery, and arbitrary gain-margin improvement of continuous-time non-minimum phase linear systems [Yan et al., 1994, Er et al., 1994, Er and Anderson, 1994].

Definition 3.1.2 (Piecewise-constant GSHF)
A piecewise-constant GSHF (PC GSHF) is given by the following pulse response:

$$h(t) = \begin{cases} a_0 & \text{if } t \in [0, T/N), \\ a_1 & \text{if } t \in [T/N, 2T/N), \\ \dots & \dots \\ a_{N-1} & \text{if } t \in [(N-1)T/N, T). \end{cases} \tag{3.11}$$

◇

The frequency response functions for FDLTI and PC GSHFs can be easily computed from their definitions, and are given by the following lemmas taken from Middleton and Freudenberg [1995].

Lemma 3.1.5 (Frequency Response Function of a FDLTI GSHF)
The frequency response function of a FDLTI GSHF defined by (3.10) *is:*

$$H(s) = K(sI + L)^{-1}(e^{LT} - e^{-sT}I)M. \tag{3.12}$$

◦

Lemma 3.1.6 (Frequency Response Function of a PC GSHF)
The frequency response function of a PC GSHF defined by (3.11) *is:*

$$H(s) = \frac{1 - e^{-sT/N}}{s} A_d(e^{-sT/N}), \tag{3.13}$$

where $A_d(z)$ *is the polynomial*

$$A_d(z) \triangleq \sum_{k=0}^{N-1} a_k z^k. \tag{3.14}$$

◦

In the rest of the chapter we shall assume that the following additional condition is satisfied by the pulse response h.

Assumption 4
The hold function h *is non-zero almost everywhere in neighborhoods of* $t = 0$ *and* $t = T$. ◦

This is a technical condition required only for simplicity of analysis; it may be removed at the expense of more complexity in the notation. This assumption may be interpreted as that the hold pulse response h has "effective" support on the whole interval $[0, T)$, e.g., no pure time-delays. This is clearly satisfied by FDLTI GSHFs, as is easily seen from (3.12). For PC GSHFs Assumption 4 is equivalent to $a_0 \neq 0 \neq a_{N-1}$.

3.2 Distribution of Zeros of GSHFs

Zeros of a hold response function have important connections with fundamental properties of the sampled-data system. For example, Middleton and Freudenberg [1995] have shown that these zeros have transmission blocking properties and can also affect the stabilizability properties of the discretized system (cf. §2.2.2 in Chapter 2). Furthermore, zeros of the hold in $\mathbb{C}^+$ impose design tradeoffs in the achievable performance of the sampled-data system, as we shall see in Chapter 4.

This section focuses on the distribution of zeros of the hold frequency response H. In Subsection 3.2.1 we describe the precise location and asymptotic distribution of the zeros of PC and FDLTI holds. Apart from the mentioned effects of "non-minimum phase" zeros on the system performance, it turns out — and we shall see it in §3.3 — that *all* zeros compromise the shape of the hold frequency response on the $j\omega$-axis. In Subsection 3.2.2, we derive a necessary condition for these GSHFs to have frequency responses with all their zeros on the $j\omega$-axis. We finish in Subsection 3.2.3 with an example that illustrates these results.

3.2.1 Zeros of PC and FDLTI GSHFs

It is difficult to make general statements about the distribution of the zeros of a GSHF. However, for important special cases, the locations and asymptotic distribution of these zeros can be described precisely. The following lemma characterizes exhaustively the zeros of PC holds, which are the GSHFs of greatest practical significance.

Lemma 3.2.1 (Zeros of a Piecewise-constant GSHF)
Consider a GSHF given by (3.11) *with associated frequency response function* H *given by* (3.13) *and* (3.14). *Then the zeros of* H *are at*

$$s = j\ell N\omega_s, \quad \textit{where } \ell = \pm 1, \pm 2, \ldots, \tag{3.15}$$

and

$$s = -\frac{N}{T}\log \xi_i + jkN\omega_s, \quad \textit{with } k = 0, \pm 1, \pm 2, \ldots, \tag{3.16}$$

where ξ_i*, with* $i = 1, 2, \ldots, N$*, is any zero of* $A_d(z)$.

Proof: From Lemma 3.1.6, H can be written as (3.13). The zeros of

$$\frac{1 - e^{-sT/N}}{s}$$

are given by (3.15). It remains, therefore, to determine the zeros of $A_d(e^{-sT/N})$, which are given precisely by (3.16). The assumption that $a_0 \neq 0$ implies that $\xi_i \neq 0$ for every i, and hence $\log \xi_i$ is defined. □

This result tells us that the zeros of a PC GSHF are essentially determined by those of the polynomial A_d, and the sampling period.

Zeros of FDLTI holds are harder to determine, but we can say something in particular cases. Consider a hold defined by (3.10), and suppose that h is not identically zero. Let m and n be the smallest nonnegative integers such that

$$h^{(m)}(0^+) \neq 0 \quad \text{and} \quad h^{(n)}(T^-) \neq 0, \tag{3.17}$$

where $h^{(k)}$ denotes the kth-derivative of h. We define

$$\eta \triangleq \frac{h^{(m)}(0^+)}{h^{(n)}(T^-)}, \tag{3.18}$$

which, for the particular case of FDLTI GSHFs, equals

$$\eta = \frac{K(-L)^m e^{LT} M}{K(-L)^n M}.$$

Then we have the following result concerning the asymptotic locations of the zeros of FDLTI GSHFs.

Lemma 3.2.2 (High Frequency Zeros of a FDLTI GSHF)
If H *is the frequency response of a FDLTI GSHF, then it has an unbounded sequence of zeros* $\{\gamma_\ell\}_{\ell=1}^{\infty}$ *"converging to infinity". Furthermore, these zeros converge to the roots of the equation* $\eta = e^{-\phi T}\phi^{n-m}$*. In particular, if* $n = m$*, the zeros converge to the sequence defined by*

$$\phi_\ell = -\frac{1}{T}\log\eta + j\ell\omega_s, \quad \ell = 0, \pm1, \pm2, \dots \tag{3.19}$$

Proof: See §A.2 in Appendix A. □

A precise description of the zeros of a FDLTI hold is possible in a particular case, as we see in the following lemma.

Lemma 3.2.3 (Zeros of a FDLTI GSHF (Special Case))
Consider a FDLTI GSHF, and suppose that $KM \neq 0$*. Assume that* $L = \lambda I$*, where* I *is the identity matrix and* λ *is a scalar. Then the zeros of* H *are located precisely at*

$$\gamma_\ell = -\lambda + j\ell\omega_s, \quad \ell = \pm1, \pm2, \dots \tag{3.20}$$

Proof: Since $KM \neq 0$, H is not identically zero. The special structure of L implies that

$$H(s) = KM\frac{e^{\lambda T} - e^{-sT}}{s+\lambda},$$

and the result follows. □

Remark 3.2.1 (Approximation of the zeros of a FDLTI GSHF) Notice that since a FDLTI GSHF will most probably be implemented as a PC GSHF, the additional difficulty in characterizing zeros of FDLTI holds over PC holds is somehow deprived of practical significance[2].

◇

3.2.2 GSHFs with all Zeros on the $j\omega$-axis

A well-known property of a hybrid control system using a ZOH in conjunction with a discrete integrator is the ability to asymptotically reject step disturbances. This arises from the fact that the ZOH frequency response has zeros at multiples of the sampling frequency $\omega_s = 2\pi/T$ on the $j\omega$-axis,

$$H_{ZOH}(jk\omega_s) = 0, \quad \text{for } k = \pm1, \pm2, \dots.$$

[2] See the example in Subsection 3.2.3.

In addition, these zeros contribute to diminish high frequency components of the plant response that are aliased down to low frequencies. This is particularly important in sampled-data control applications, where the low-frequency range is typically of great interest.

The response of a GSHF, on the other hand, need not have zeros at these frequencies, and thus high frequency plant behavior (and uncertainty) may have significant effect on the low-frequency range of the hybrid control system [cf. Feuer and Goodwin, 1994]. To get a preliminary intuitive view of this, compare for example the GSHF response with the response of a ZOH, plotted in Figure 3.5; this GSHF is taken from Kabamba [1987, Example 2].

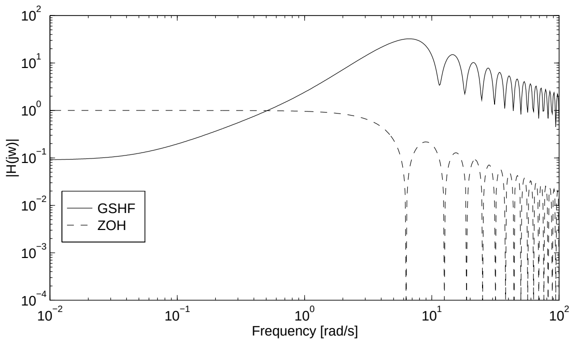

Figure 3.5: Frequency response of hold functions.

In addition, zeros of H close to unstable open-loop poles of the plant may render an ill-conditioned discrete-time system Middleton and Freudenberg [1995], Middleton and Xie [1995], due to an approximate pole-zero cancelation that tend to violate the non-pathological sampling assumption of Lemma 2.2.1. Moreover, as will become clear in §3.3, also zeros in $\overline{\mathbb{C}^-}$ compromise the frequency response of the hold, depending on the specifications that this frequency response is required to meet.

An interesting question then arises from the above observations: What is the class of GSHFs that, as the ZOH, have all their zeros on the $j\omega$-axis? The following proposition gives a necessary condition that the hold frequency response must satisfy to have such a zero distribution.

Proposition 3.2.4
Let H *be the frequency response function of a PC or a FDLTI GSHF; suppose that* h *satisfy Assumption 4. Then if* H *has all its zeros on the* $j\omega$*-axis, either*

$$\mathsf{H}(s) = e^{-sT}\,\mathsf{H}(-s), \tag{3.21}$$

or

$$\mathsf{H}(s) = -e^{-sT}\,\mathsf{H}(-s). \tag{3.22}$$

Proof: Suppose that $\{ja_k\}$ are the nonzero zeros of H repeated according to multiplicity, and that H has a zero at $z = 0$ of order $p \geq 0$ ($p = 0$ means that $H(0) \neq 0$). Since H is an entire function of exponential type T, using the Hadamard Factorization Theorem [e.g., Markushevich, 1965] we can represent it as

$$H(s) = s^p e^{g_0 + g_1 s} \prod_{k=1}^{\infty} \left(1 - \frac{s}{ja_k}\right) e^{s/ja_k}, \tag{3.23}$$

where g_0 and g_1 are real numbers. Without lost of generality we may assume $g_0 = 0$ (since otherwise we consider $H(s)e^{-g_0}$), and since the zeros are symmetric with respect to the real axis, (3.23) simplifies to

$$H(s) = s^p e^{g_1 s} \prod_{\ell=1}^{\infty} \left(1 + \frac{s^2}{a_\ell^2}\right), \tag{3.24}$$

where now $\{a_\ell\}$ denote the zeros in the upper (or lower) half of the $j\omega$-axis.

As in Subsection 3.2.1 let m and n be the smallest integers such that (3.17) holds. Notice that both $h^{(m)}(0^+)$ and $h^{(n)}(T^+)$ are nonzero finite numbers for PC and FDLTI GSHFs with compact support on $[0, T)$. Hence, the number η defined in (3.18) is also nonzero and finite. Next we use the Initial Value Theorem [e.g., Zemanian, 1965] to compute $h^{(n)}(0^+)$ from (3.24). Thus, for x real we have that

$$\begin{aligned} h^{(m)}(0^+) &= \lim_{x \to \infty} x^{m+1} H(x) \\ &= \lim_{x \to \infty} x^{p+m+1} e^{g_1 x} \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{a_\ell^2}\right). \end{aligned} \tag{3.25}$$

An analogous expression can be obtained for $h^{(n)}(T^-)$ following similar steps with $H(-s)e^{-sT}$,

$$\begin{aligned} h^{(n)}(T^-) &= \lim_{x \to \infty} (-x)^{n+1} e^{-xT} H(-x) \\ &= \lim_{x \to \infty} (-1)^{n+p} x^{p+n+1} e^{-(g_1+T)x} \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{a_\ell^2}\right). \end{aligned} \tag{3.26}$$

Therefore, we can write from (3.25) and (3.26),

$$\begin{aligned} \eta &= \frac{h^{(m)}(0^+)}{h^{(n)}(T^-)} \\ &= \lim_{x \to \infty} (-1)^{n+p} x^{m-n} e^{(2g_1+T)x}. \end{aligned} \tag{3.27}$$

Since η is nonzero and finite, it necessarily follows from (3.27) that $m = n$ and $g_1 = -T/2$. With this value of g_1 in (3.24), it is easy to check that H verifies the required conditions (3.21) or (3.22) (the sign depending on the order of the zero at $s = 0$), completing the proof. □

Notice in the proof above that the conditions $m = n$, and $g_1 = -T/2$ imply that $\eta = (-1)^{n+p}$, which in turn, by Lemma 3.2.2, tells us that the zeros of H approach asymptotically to the $j\omega$-axis as the distance from the origin increases. We could say then that conditions (3.21) and (3.22) become also "sufficient" for large values of s.

The fact that $\eta = (-1)^{n+p}$ also suggests that if H has all its zeros on the $j\omega$-axis, then h has some kind of symmetry with respect to the middle point of the interval $[0, T)$. For example, if $n = 1$ and $p = 0$ say, then $h(0^+) = 0 = h(T^-)$, and the corresponding derivatives are mirrored, $h'(0^+) = -h'(T^-)$. In fact, conditions (3.21) and (3.22) are *equivalent* to "symmetry" of h, as we shall prove next. Let us first make more precise what we mean by this.

Definition 3.2.1 (Symmetry of h)
We say that h has *even* (*odd*) symmetry if $h(t) = h(T - t)$ ($h(t) = -h(T - t)$). We say that h is *symmetric* if h has either even or odd symmetry. $\diamond$

The following corollary to Proposition 3.2.4 establishes that holds with all their zeros on the $j\omega$-axis are necessarily symmetric in the sense just defined.

Corollary 3.2.5
If H has all its zeros on the $j\omega$-axis, then h is symmetric. Moreover,

(i) if H has none or an even number of zeros at $s = 0$, then h is even symmetric;

(ii) if H has an odd number of zeros at $s = 0$, then h is odd symmetric.

Proof: We prove only (i); the proof of (ii) is obtained in a similar way. We know from Proposition 3.2.4 that if H has all its zeros on the $j\omega$-axis — and none or an even number of them at $s = 0$, then condition (3.21) is satisfied. Write the Fourier Series representation of h,

$$h(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{jk\omega_s t}, \tag{3.28}$$

with

$$\begin{aligned} c_k &= \frac{1}{T}\int_0^T e^{-jk\omega_s t}\, h(t)\, dt \\ &= \frac{H(jk\omega_s)}{T}. \end{aligned}$$

We prove now that condition (3.21) is satisfied if and only if all c_k — i.e., $H(jk\omega_s)$ — are real. Indeed, if c_k is real for all k, then $H(jk\omega_s) = H(-jk\omega_s)$, and using

Lemma 3.1.3 we have that

$$\begin{aligned}
H(s) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} H(jk\omega_s)\, H_{ZOH}(s - jk\omega_s) \\
&= \frac{1}{T}\sum_{\ell=-\infty}^{\infty} H(j\ell\omega_s)\, H_{ZOH}(s + j\ell\omega_s) \\
&= \frac{e^{-sT}}{T}\sum_{\ell=-\infty}^{\infty} H(j\ell\omega_s)\, H_{ZOH}(-s - j\ell\omega_s) \\
&= e^{-sT}H(-s).
\end{aligned}$$

The converse is immediate. Finally, (i) follows easily from noting that h has all real Fourier coefficients if and only if it is even symmetric. □

This corollary provides an easy way of determining whether the hold frequency response H may have zeros off the $j\omega$-axis by just examining the shape of the hold function h; i.e., an asymmetric h will necessary imply zeros in either $\mathbb{C}^+$ or $\mathbb{C}^-$.

Remark 3.2.2 (A Conjecture for a General GSHF) Strictly, we have proved these results only for PC and FDLTI GSHFs; nevertheless, we could conjecture that they hold for the general case. Indeed, notice that any admissible h can be arbitrarily approximated by a piecewise-constant function h_{PC}, and then the zeros of H_{PC} will approximate the zeros of H. Then, it is clear that if H has all its zeros on the $j\omega$-axis, we can build a sequence of symmetric PC functions whose zeros will approach to the $j\omega$-axis. Since the result holds for H_{PC}, we can expect that, in the limit, it will hold also for H. A rigorous proof seems difficult, though. ◇

3.2.3 Example: Zeros of a FDLTI GSHF

In this example we illustrate the previous results on zero locations of GSHFs. We consider the FDLTI GSHF used in Kabamba [1987, Example 2] to simultaneously stabilize two continuous-time systems that violate the appropriate parity interlacing property (cf. Example 7.1.1). The matrices that define the hold function with the characterization given by Definition 3.1.1 are

$$K = \begin{bmatrix}1 & 0 & 0\end{bmatrix}, \quad L = \begin{bmatrix}1 & 1 & 0\\ 2 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}, \quad M = \begin{bmatrix}87.5619\\ -616.4937\\ 1322.6\end{bmatrix}.$$

The sampling period is $T = 1$s, to which corresponds a sampling frequency $\omega_s = 2\pi$. The hold function h is shown on the left in Figure 3.6.

As we can see in the figure, h is not symmetric, so we know by Corollary 3.2.5 that the corresponding frequency response H *will* have zeros off the $j\omega$-axis. Indeed, this can be seen in Figure 3.7, where we have plotted a section of the function $\log|H|$ for $s = x + jy$, with $-1.5 \leq x \leq 0.5$ and $-\omega_s/2 \leq y \leq 3\omega_s$. The zeros

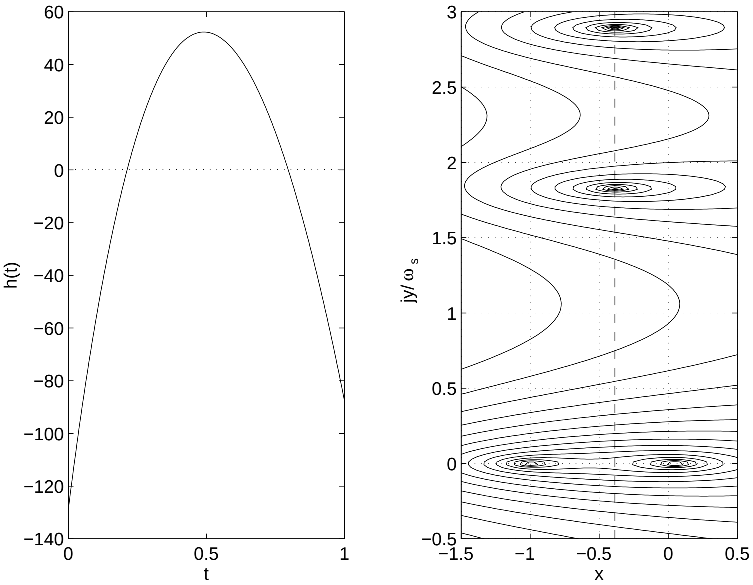

Figure 3.6: Hold function h, and contour plot of H.

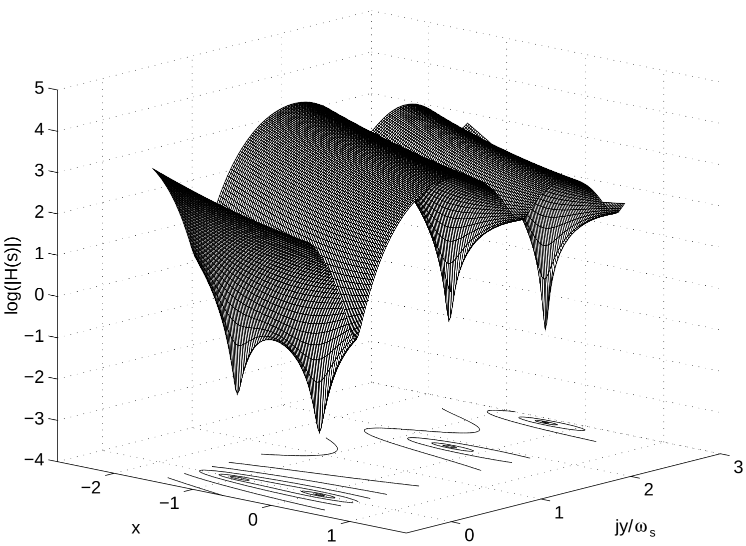

Figure 3.7: Zeros of H.

are indicated by the negative peaks on this surface; a contour plot is given on the right in Figure 3.6.

Let us check the asymptotic behavior of the zeros of H. Following the notation of Lemma 3.2.2 we compute η from (3.18), which, for the values of K, L, and M given is $\eta = 1.4718$, with $n = 0 = m$. Then, by Lemma 3.2.2 we know that there is an infinite sequence of zeros that approach asymptotically to a sequence given by (3.19), which for this case is

$$\phi_\ell = -0.3865 + j\ell\omega_s. \tag{3.29}$$

This is can be anticipated already in Figure 3.6, where for reference we have drawn a vertical line at $x = -0.3865$.

As we discussed in Remark 3.2.1, the zeros of a FDLTI GSHF can be approximated by the zeros of a PC GSHF, which are completely characterized in Lemma 3.2.1. This is verified in Figures 3.8, 3.9, and 3.10, where we have depicted analogs to Figure 3.6 for PC approximations to h with 4, 32, and 256 partitions respectively. We can see there how the zeros approach to the locations given in Figure 3.6 as the number of partitions is increased. Notice, however, that the convergence is slow, particularly for those zeros on the real axis.

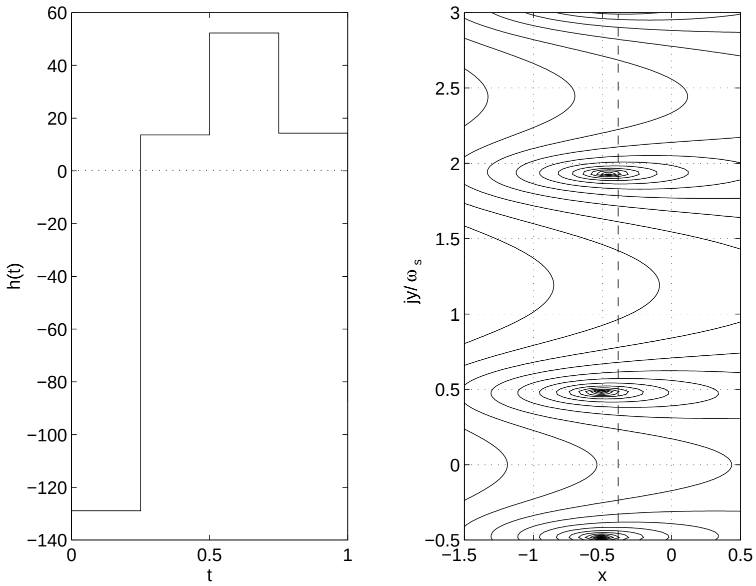

Figure 3.8: PC approximation to h, $N = 4$.

3.3 Integral Relations

Integral relations quantifying inherent limitations in the achievable performance of feedback control systems have been known for some time. Bode and Poisson

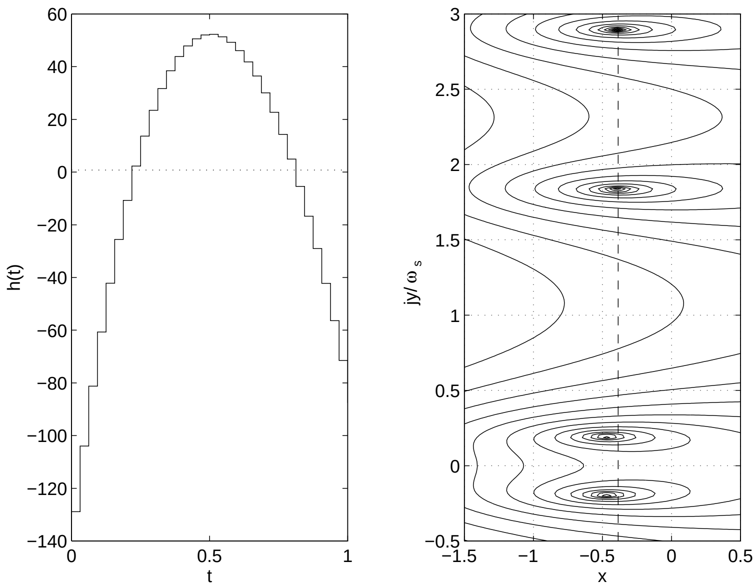

Figure 3.9: PC approximation to h, N = 32.

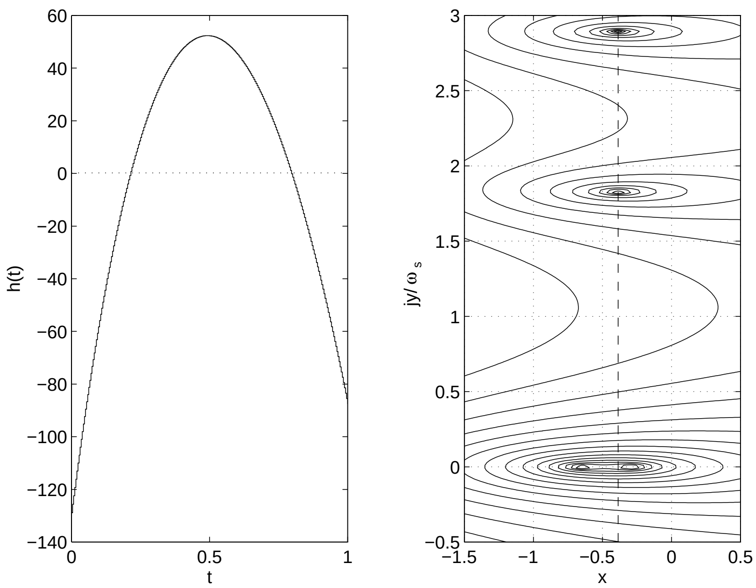

Figure 3.10: PC approximation to h, N = 256.

integrals on the sensitivity and complementary sensitivity functions of continuous-time systems have been used to describe design tradeoffs arising from the feedback structure, and the requirement of closed-loop stability Freudenberg and Looze [1985]. Similar results have been also extended to discrete-time systems Sung and Hara [1988], Middleton and Goodwin [1990]. In Chapter 4 we shall study the case of sampled-data systems.

In this section, we present two types of integral relations for the frequency response of a GSHF. These integrals translate the connection between growth characteristics and the distribution of zeros into constraints that the magnitude of the frequency response of the GSHF must satisfy on the $j\omega$-axis. We show how zeros off the $j\omega$-axis impose tradeoffs over the values of the frequency response of the hold at all frequencies. In contrast, a zero *on* the $j\omega$-axis only imposes a constraint at one point (namely, the frequency response is zero at that frequency).

3.3.1 Poisson Integral for GSHFs

Let H be the frequency response function of a GSHF defined by (2.4). Recall the definition of the Poisson kernel for the half plane, [e.g., Freudenberg and Looze, 1988]. Let $s = x + jy$, s in $\mathbb{C}$, and let ω be a real number. We define the Poisson kernel for the half plane, $\Psi(s, \omega)$, by

$$\Psi(s,\omega) \triangleq \frac{x}{x^2 + (\omega - y)^2} + \frac{x}{x^2 + (\omega + y)^2}. \tag{3.30}$$

The following proposition presents a Poisson integral relation for the function $1 - H$.

Proposition 3.3.1 (Poisson integral for $1 - H$)
Let $\xi = x + jy$ be a zero of H. Then

(i) if $x > 0$,

$$\int_0^\infty \log|1 - H(j\omega)|\,\Psi(\xi,\omega)\,d\omega \geq 0; \tag{3.31}$$

(ii) if $x < 0$,

$$\int_0^\infty \log|1 - H(j\omega)\,e^{j\omega T}|\,\Psi(-\xi,\omega)\,d\omega \geq 0. \tag{3.32}$$

Proof: We prove only (3.31); (3.32) is obtained in a similar way. Consider the function $F = 1 - H$. Since H is entire, so is F, and therefore, we may factorize it as

$$F = \tilde{F}\,B,$$

where $\tilde{F}$ is an entire function without zeros in $\mathbb{C}^+$, and B is the Blaschke product of the sequence of zeros of F, $\{\phi_k\}_{k=1}^{N_\phi}$ (with N_ϕ possibly infinite), in $\mathbb{C}^+$,

$$B = \prod_{k=1}^{N_\phi} \frac{\phi_k - s}{\overline{\phi}_k + s}.$$

Note that $\log \tilde{F}$ is analytic in $\mathbb{C}^+$, and furthermore, it satisfies the conditions for a Poisson Integral representation Freudenberg and Looze [1988]. Hence, we can write,

$$\begin{aligned}\int_0^\infty \log|\tilde{F}(j\omega)|\,\Psi(s,\omega)\,d\omega &= \pi \log|\tilde{F}(s)| \\ &= \pi \log|F(s)| - \pi \log|B(s)|. \qquad (3.33)\end{aligned}$$

Evaluating (3.33) at a zero of H in $\mathbb{C}^+$, and noting that $|\tilde{F}(j\omega)| = |F(j\omega)|$ and $-\log|B(s)| \geq 0$ for each s in $\mathbb{C}^+$, we get inequality (3.31), completing the proof. Inequality (3.32) is obtained similarly by starting with the function $F(s) = 1 - H(-s)\,e^{-sT}$. □

It follows from (3.31) and (3.32) that zeros of the hold off the $j\omega$-axis impose design tradeoffs on its frequency response. More specifically, if we require that $|H(j\omega)|$ be close to 1 over some range of frequencies, $|H(j\omega)|$ will necessarily show a peak somewhere else. The extent of this difficulty is linked to the relative location of these zeros, and depends on the specifications that the hold frequency response is required to satisfy on the low-frequency range, as we see next.

Consider an interval of low frequencies $\Omega = [0, \omega_b]$, where $\omega_b \leq \omega_s/2$, and suppose that we require the hold response $H(j\omega)$ to be close to 1 over this interval. The interval Ω may be interpreted as the closed-loop bandwidth of a hybrid feedback system with hold H. Asking $H(j\omega)$ not too large on Ω is a reasonable specification in practice, since a "high gain" hold may bring in difficulties with actuator saturations (cf. Lemma 3.1.2 and the discussion following). We state this requirement as

$$|1 - H(j\omega)| < \alpha, \qquad \text{for } \omega \text{ in } \Omega = [0, \omega_b], \tag{3.34}$$

where α is a small positive number.

Assume that H has a zero $\xi = x + jy$ in either $\mathbb{C}^+$ or $\mathbb{C}^-$. Let $\Theta(\xi, \Omega)$ denote the weighted length of the interval Ω with the Poisson kernel for the half plane,

$$\Theta(\xi, \Omega) \triangleq \int_0^{\omega_b} \Psi(\xi, \omega)\,d\omega, \tag{3.35}$$

It is not difficult to check that

$$\Theta(\xi, \Omega) = \arctan\left(\frac{\omega_b - y}{x}\right) + \arctan\left(\frac{\omega_b + y}{x}\right).$$

As discussed in Freudenberg and Looze [1985] and Freudenberg and Looze [1988], the weighted length $\Theta(\xi, \Omega)$ may also be interpreted as a phase lag introduced by the term of a Blaschke product corresponding to the zero ξ over Ω; e.g., if ξ is real, then

$$\Theta(\xi, \Omega) = -\sphericalangle \frac{\xi - j\omega_s}{\xi + j\omega_s}\ ; \tag{3.36}$$

i.e., the weighted length of the interval Ω equals the negative of the phase lag contributed by the Blaschke product $(\xi - s)/(\xi + s)$ at the upper end point of the interval.

The following result is a straightforward consequence of Proposition 3.3.1.

Corollary 3.3.2
Suppose that $|H(j\omega)|$ *satisfies* (3.34). *Then, if* $\xi = x + jy$ *is a zero of* H, *with* $x \neq 0$,

(i) if $x > 0$,

$$\sup_{\omega > \omega_b} |H(j\omega)| \geq \left(\frac{1}{\alpha}\right)^{M_\Omega(\xi)} - 1, \tag{3.37}$$

(ii) if $x < 0$,

$$\sup_{\omega > \omega_b} |H(j\omega)| \geq \left(\frac{1}{\alpha}\right)^{M_\Omega(-\xi)} - 1, \tag{3.38}$$

where

$$M_\Omega(\xi) = \frac{\Theta(\xi, \Omega)}{\pi - \Theta(\xi, \Omega)}$$

∘

Note from (3.37) and (3.38), that the effect of zeros in $\mathbb{C}^-$ is as detrimental as the effect of zeros in $\mathbb{C}^+$. To illustrate these bounds, suppose that we wish to design a hold satisfying specification (3.34), and that due to the type of hold chosen, there will be a real zero $\xi = x$. It follows that if the hold response has zeros off the $j\omega$-axis, and we require $|H(j\omega)|$ very close to 1 on Ω, then $|H(j\omega)|$ will necessarily have a large peak at higher frequencies. The tradeoff relaxes as the zero tends to be located at relatively high frequencies. By contrast, if the zero gets closer to the $j\omega$-axis, the constraint worsen. In the limit, when the zero is *on* the $j\omega$-axis, then the Poisson integrals (3.31) and (3.32) collapse into an algebraic constraint[3]: $0 \leq \log|1 - H(\xi)| = 0$. Figure 3.11 shows plots of the bounds (3.37) and (3.38) vs. the location of the zero in Ω for different values of the specification α.

3.3.2 Middleton Integral for GSHFs

Another integral relation that evidences the penalties imposed by zeros and poles of an analytic function over its values on the $j\omega$-axis has been proposed by Middleton and Goodwin [1990, Corollary 13.4.1]. This integral relation can be used to quantify the effect of zeros off the $j\omega$-axis of the frequency response of holds with $H(0) \neq 0$. In contrast with the previous integral relation, here we obtain a single integral for both cases of zeros in $\mathbb{C}^-$ and $\mathbb{C}^+$.

Denote by $\{\lambda_k\}_{k=1}^{N_\lambda}$ and $\{\rho_k\}_{k=1}^{N_\rho}$ the sequence of zeros of H, counted with multiplicities, in $\mathbb{C}^-$ and $\mathbb{C}^+$, respectively. Typically, $N_\lambda + N_\rho = \infty$. Without lost of generality, we also assume $H(0) = 1$. Then, we have the following result.

[3]For which the specification (3.34) becomes incompatible unless $\alpha \geq 1$.

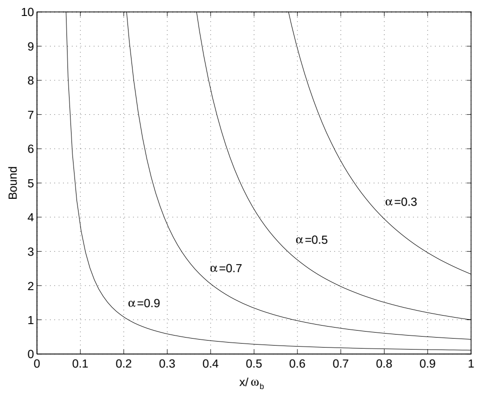

Figure 3.11: Lower bound (3.37) (or (3.38)) for a real zero.

Proposition 3.3.3 (Middleton Integral for H**)**

$$\int_0^\infty \frac{\log|H(j\omega)|}{\omega^2}\,d\omega = \frac{\pi}{4}\left(-T - 2\sum_{k=1}^{N_\lambda}\frac{1}{\lambda_k} + 2\sum_{k=1}^{N_\rho}\frac{1}{\rho_k}\right). \tag{3.39}$$

Proof: Factorize H as $H = \tilde{H}\,B_\rho$, where B_ρ is the Blaschke product of the zeros of H in $\mathbb{C}^+$,

$$B_\rho(s) = \prod_{k=1}^{N_\rho}\frac{\rho_k - s}{\overline{\rho}_k + s}.$$

From similar arguments to those in the proof of Proposition 3.3.1, we obtain the following Poisson Integral relation, which we evaluate at a real $s = x$, $x > 0$,

$$\int_0^\infty \log|H(j\omega)|\,\frac{2x}{x^2+\omega^2}\,d\omega = \pi\log|H(x)| - \pi\log|B_\rho(x)|. \tag{3.40}$$

Dividing both sides of (3.40) by x, and taking the limit when $x \to 0$ yields[4].

$$2\int_0^\infty \frac{\log|H(j\omega)|}{\omega^2}\,d\omega = \lim_{x\to 0}\pi\frac{\log|H(x)|}{x} - \lim_{x\to 0}\pi\frac{\log|B_\rho(x)|}{x}. \tag{3.41}$$

The application of L'Hopital's rule to the limits on the RHS of (3.41), our assumption $H(0) = 1$, and the fact that zeros of H must occur in complex conjugate pairs, yield

$$2\int_0^\infty \frac{\log|H(j\omega)|}{\omega^2}\,d\omega = \left.\frac{dH(s)}{ds}\right|_{s=0} + 2\sum_{k=1}^{N_\rho}\frac{1}{\rho_k}. \tag{3.42}$$

[4]The interchange between limit and integration on the LHS is valid by the Lebesgue Dominated Convergence Theorem Riesz and Sz.-Nagy [1990]

From analogous arguments applied to $H(-s)e^{-sT}$, and noting that $|H(j\omega)| = |H(-j\omega)|$, we obtain an integral relation for the zeros of H in $\mathbb{C}^-$,

$$2\int_0^\infty \frac{\log|H(j\omega)|}{\omega^2}\,d\omega = -T - \left.\frac{dH(s)}{ds}\right|_{s=0} - 2\sum_{k=1}^{N_\lambda}\frac{1}{\lambda_k}. \tag{3.43}$$

Finally, adding term-to-term (3.42) and (3.43) yields (3.39), completing the proof. □

In the particular case of a PC hold, the location of zeros is well determined, and therefore, we obtain a more specific result. As it follows from Lemma 3.2.1, zeros off the $j\omega$-axis for a PC hold are determined by the zeros of the discrete polynomial $A_d(z)$. Denote by $\{\phi_k\}_{k=1}^{N_\phi}$ and $\{\psi_k\}_{k=1}^{N_\psi}$ the set of zeros of $A_d(z)$ inside and outside the unit circle, respectively (note that there is a finite number of them, $N_\psi + N_\phi \le N$). The following corollary states the analog to Proposition 3.3.3 for PC holds.

Corollary 3.3.4 (Middleton Integral for PC GSHFs)

$$\int_0^\infty \frac{\log|H(j\omega)|}{\omega^2}\,d\omega = \frac{\pi T}{4}\left(\frac{1}{N}\sum_{k=1}^{N_\psi}\frac{\psi_k+1}{1-\psi_k} + \frac{1}{N}\sum_{k=1}^{N_\phi}\frac{\phi_k+1}{\phi_k-1} - 1\right). \tag{3.44}$$

Proof: From Lemma 3.2.1 we have that zeros of H in $\mathbb{C}^+$ are

$$\rho_{k,i} = -\frac{N}{T}\log\phi_i + jkN\omega_s, \quad \text{with } i = 1,\ldots,N_\phi \text{ and } k = 0,\pm1,\pm2\ldots \tag{3.45}$$

and zeros in $\mathbb{C}^-$ are

$$\lambda_{k,i} = -\frac{N}{T}\log\psi_i + jkN\omega_s, \quad \text{with } i = 1,\ldots,N_\psi \text{ and } k = 0,\pm1,\pm2\ldots, \tag{3.46}$$

From (3.45) and (3.46), and using the identity [Rudin, 1987, p. 195]

$$\frac{e^{2\pi x}+1}{e^{2\pi x}-1} = \frac{1}{\pi}\sum_{k=-\infty}^{\infty}\frac{x}{x^2+k^2},$$

in the form

$$\frac{e^{2\pi\alpha/\beta}+1}{e^{2\pi\alpha/\beta}-1} = \frac{\beta}{\pi}\sum_{k=-\infty}^{\infty}\frac{1}{\alpha+jk\beta},$$

we obtain the following closed forms for the series on the RHS of (3.39),

$$\sum_{i=1}^{N_\phi}\sum_{k=-\infty}^{\infty}\frac{1}{\lambda_{k,i}} = \frac{T}{2N}\sum_{i=1}^{N_\phi}\frac{\phi_k+1}{\phi_k-1},$$

and

$$\sum_{i=1}^{N_\psi}\sum_{k=-\infty}^{\infty}\frac{1}{\rho_{k,i}} = \frac{T}{2N}\sum_{i=1}^{N_\psi}\frac{\psi_k+1}{\psi_k-1}.$$

Replacing these in (3.39) yields (3.44), concluding the proof. □

Proposition 3.3.3 and Corollary 3.3.4 show that if $|H(j\omega)| < 1$ over some frequency range, then it must necessarily be greater than one at other frequencies. This tradeoff is minimized if H has all its zeros on the $j\omega$-axis, and can only worsen if there are zeros off the $j\omega$-axis. Indeed, notice on the RHSs of (3.39) and (3.44) that the terms due to zeros off the $j\omega$-axis are always positive. In particular, the RHSs of (3.39) and (3.44) can get arbitrarily large with real zeros of H approaching $s = 0$.

Remark 3.3.1 (Middleton Integral and Zero Density) There is an interesting connection between this integral relation and the results of §3.2. Let $\{a_k\}$ denote the infinite sequence of zeros of H repeated according multiplicity. If $n(r)$ denotes the number of zeros of moduli not exceeding r, i.e., the number of a_k such that $|a_k| < r$, with r real positive, we define the *density of zeros* δ as

$$\delta \triangleq \lim_{r\to\infty} \frac{n(r)}{r}.$$

The following is a well-known result for entire functions of exponential type.

Proposition 3.3.5 (Boas [1954, Theorem 8.2.1])
If H is an entire function of exponential type with all its zeros on the $j\omega$-axis, then the following two conditions

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{|H(j\omega)|}{\omega^2}\, d\omega = -\pi^2 B, \tag{3.47}$$

and

$$\lim_{r\to\infty} \frac{n(r)}{r} = 2B \tag{3.48}$$

are equivalent. ○

If we consider the integral relation (3.39) for a GSHF without zeros off the $j\omega$-axis, we obtain

$$\int_0^{\infty} \frac{\log|H(j\omega)|}{\omega^2}\, d\omega = -\frac{\pi T}{4}. \tag{3.49}$$

Since $|H(j\omega)| = |H(-j\omega)|$ we can change the interval of integration on the LHS of (3.49) to $(-\infty, \infty)$ by multiplying by 2 its RHS. Then, according to (3.47) we have that $B = \omega_s^{-1}$, and therefore by Proposition 3.3.5 the density of zeros of H is

$$\delta = 2/\omega_s.$$

From this we can deduce that the number of zeros of H in a ball of radius $r = k\omega_s$ is approximately $2k$ when k is large, in agreement with our previous result of Lemma 3.2.2. ◇

3.3.3 Example: Tradeoffs in $H(j\omega)$

To illustrate the above results, we take an example from Er and Anderson [1994], where a PC hold of two steps is used to achieve discrete-time perfect loop transfer recovery of a non-minimum phase continuous plant. The zero-placement capabilities of GSHFs are used in their algorithm. The PC hold obtained for a sampling time $T = 0.04s$, is the following:

$$h(t) = \begin{cases} -1957 & \text{for } 0 \le t < 0.02 \\ 1707 & \text{for } 0.02 \le t < 0.04 \end{cases} \tag{3.50}$$

From Lemma 3.2.1 we see that there is an infinite sequence of zeros at

$$s = -6.8338 + jk\omega_s, \quad k = 0, \pm 1, \pm 2, \ldots,$$

with $\omega_s = 157.0796$. Figure 3.12 shows the normalized magnitude of $H(j\omega)$ (such that $H(0) = 1$), compared to that of a ZOH. We can see that the frequency response of this GSHF displays large peaks both within and outside the Nyquist range of frequencies, peaks that will tend to amplify potential plant uncertainties and disturbances at those frequencies. The magnitude of these peaks may be estimated by considering the bound (3.38). In this case, the bandwidth of the closed loop system is given [from Er and Anderson, 1994] by $\omega_b = 15.3\text{rad/s}$, so we have that the ratio $x/\omega_b = 0.44$. This value in Figure 3.11 gives an indication of the peak expected in $|H(j\omega)|$ when we require that the specification (3.34) be satisfied on the interval $[0, \omega_b]$. $\diamond$

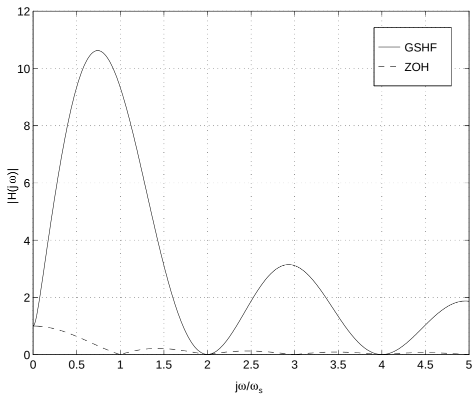

Figure 3.12: GSHF frequency response.

3.4 Summary

In this chapter, we have analyzed the frequency response and zero locations of non-traditional D-A devices known as GSHFs. We have presented general properties and results concerning norms and boundary values arising from the fact that the frequency response of a GSHF is an entire function. In particular, we have shown that a frequency response with a large infinity-norm implies a hold device with large BIBO gain, which may bring in implementation difficulties in conjunction with the presence of plant input saturation.

Differently to the ZOH, GSHFs may have zeros off the $j\omega$-axis, which — as we shall see in detail in the following chapter — may seriously damage sensitivity and robustness properties of the hybrid system. For two important classes of GSHFs, we have obtained exact and asymptotic characterizations of their zero locations in function of the hold response data. A key result of this chapter shows that if the hold has all its zeros on the $j\omega$-axis, then necessarily its pulse response function has to meet certain condition of symmetry on its interval of definition.

In addition, we have derived integral relations displaying the connections between zero locations and the frequency response of the hold. Our results indicate there exist design tradeoffs that may imply frequency responses with relatively large values at high frequencies. This again, may be detriment on the sensitivity and robustness characteristics of the hybrid system, since these large values will amplify the effects of high frequency plant behavior and uncertainty on lower frequencies.