

# Generalized $S$ -Procedure and Finite Frequency KYP Lemma

TETSUYA IWASAKI <sup>a,\*</sup>, GJERRIT MEINSMA <sup>b,†</sup> and MINYUE FU <sup>c,‡</sup>

<sup>a</sup>*Department of Control Systems Engineering, Tokyo Institute of Technology, 2-12-1 Oookayama, Meguro, Tokyo 152, Japan;* <sup>b</sup>*Department of Systems, Signals and Control, Faculty of Mathematical Sciences, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands;* <sup>c</sup>*Department of Electrical and Computer Engineering, University of Newcastle, Callaghan, NSW 2308, Australia*

(Received 14 December 1998)

The contribution of this paper is twofold. First we give a generalization of the  $S$ -procedure which has been proven useful for robustness analysis of control systems. We then apply the generalized  $S$ -procedure to derive an extension of the Kalman–Yakubovich–Popov lemma that converts a frequency domain condition within a finite interval to a linear matrix inequality condition suitable for numerical computations.

**Keywords:** Control systems;  $S$ -procedure; Positive-real lemma

## 1 INTRODUCTION

Consider the following condition given by multiple inequalities:

$$\zeta^* \Theta \zeta < 0, \quad \forall \zeta \in \mathcal{G}, \quad (1)$$

$$\mathcal{G} := \{\zeta \in \mathbb{C}^n: \zeta \neq 0, \zeta^* S_i \zeta \leq 0, \forall i = 1, \dots, m\}, \quad (2)$$

---

\* Corresponding author. E-mail: iwasaki@ctrl.titech.ac.jp.

† E-mail: g.meinsma@math.utwente.nl.

‡ E-mail: eemf@ee.newcastle.edu.au.

where  $\Theta$  and  $S_i$  are given Hermitian matrices. It is trivial to verify that a sufficient condition for (1) is given by

$$\exists \tau_i > 0 \quad \text{such that} \quad \Theta < \sum_{i=1}^m \tau_i S_i. \quad (3)$$

The  $S$ -procedure [1] is to replace the multiple inequality constraint in (1) by the single inequality in (3) with multipliers  $\tau_i$ . While this procedure is concerned with the quadratic forms on  $\mathbb{C}^n$ , an extension is available [2] to the case of the quadratic forms on  $\mathcal{L}_2$ , the set of square integrable vector-valued functions.

In general, the  $S$ -procedure on  $\mathbb{C}^n$  is conservative, i.e. (3) is only sufficient for (1) and may not be necessary. Nevertheless, the condition (3) can be efficiently verified by searching for the parameters  $\tau_i$  which is a finite dimensional convex feasibility problem. Indeed, the  $S$ -procedure and the aforementioned extension have been shown to be useful for developing various methods for control systems analysis and synthesis [2–4].

When applying the  $S$ -procedure, the main concern is whether or not the procedure is conservative for the particular condition at hand. This fact gives rise to the following fundamental question: When does the  $S$ -procedure yield an exact (nonconservative) condition? This question has already been extensively studied by Yakubovich and others. It is shown (for the nonstrict inequality case) that the  $S$ -procedure on  $\mathbb{C}^n$  is exact if  $m \leq 2$  and that for  $m > 2$  there are  $\Theta$  and  $S_i$  such that the  $S$ -procedure is conservative [1,4,5]. Moreover, the  $S$ -procedure on  $\mathcal{L}_2$  is known to be exact regardless of the number of constraints  $m$  [2].

In this paper, we generalize the  $S$ -procedure on  $\mathbb{C}^n$  in the following manner: note that the set  $\mathcal{G}$  in (2) can be characterized by

$$\mathcal{G} = \{\zeta \in \mathbb{C}^n: \zeta \neq 0, \zeta^* S \zeta \leq 0, \forall S \in \mathcal{S}\} \quad (4)$$

where

$$\mathcal{S} := \left\{ \sum_{i=1}^m \tau_i S_i: \tau_i > 0, \forall i = 1, \dots, m \right\}.$$

Then the  $S$ -procedure is to replace condition (1), defined together with (4), by the existence of  $S \in \mathcal{S}$  such that  $\Theta < S$ . Now, if we consider a

general class of matrices  $\mathcal{S}$  instead of the one given above, the  $S$ -procedure is still valid, i.e. the latter condition is sufficient to guarantee (1). We call this the generalized  $S$ -procedure.

The first contribution of this paper is to show conditions on  $\mathcal{S}$  under which the generalized  $S$ -procedure is exact, and give a specific set  $\mathcal{S}$  that satisfies the conditions. The second contribution is to show that the celebrated Kalman–Yakubovich–Popov (KYP) lemma [6,7] and its extension to the finite frequency condition simply follow from the generalized  $S$ -procedure. The finite frequency KYP lemma thus obtained is useful for solving various control problems including the integrated design of dynamical systems [8] and the computation of the structured singular value (upper bound) [9].

## 2 THE GENERALIZED $S$ -PROCEDURE

Let us first introduce the notion of *lossless sets*, which will turn out to be a class of  $\mathcal{S}$  in (4) leading to an *exact* (nonconservative) generalized  $S$ -procedure.

**DEFINITION 1** *A subset  $\mathcal{S}$  of  $n \times n$  Hermitian matrices is said to be lossless if it has the following properties:*

- (a)  $\mathcal{S}$  is convex.
- (b)  $S \in \mathcal{S} \Rightarrow \tau S \in \mathcal{S} \ \forall \tau > 0$ .
- (c) For each nonzero matrix  $H \in \mathbb{C}^{n \times n}$  such that

$$H = H^* \geq 0, \quad \text{tr}(SH) \leq 0 \quad \forall S \in \mathcal{S},$$

there exist vectors  $\zeta_i \in \mathbb{C}^n$  ( $i = 1, \dots, r$ ) such that

$$H = \sum_{i=1}^r \zeta_i \zeta_i^*, \quad \zeta_i^* S \zeta_i \leq 0 \quad \forall S \in \mathcal{S},$$

where  $r$  is the rank of  $H$ .

The following is one of our main results and formally states that the generalized  $S$ -procedure is exact if the set  $\mathcal{S}$  in (4) is lossless.

**THEOREM 1** (The generalized  $S$ -procedure) *Let a Hermitian matrix  $\Theta$  and a subset  $\mathcal{S}$  of Hermitian matrices be given. Suppose  $\mathcal{S}$  is lossless.*

Then the following statements are equivalent.

- (i)  $\zeta^* \Theta \zeta < 0 \ \forall \zeta \in \mathcal{G} := \{\zeta \in \mathbb{C}^n: \zeta \neq 0, \zeta^* S \zeta \leq 0 \ \forall S \in \mathcal{S}\}.$
- (ii) There exists  $S \in \mathcal{S}$  such that  $\Theta < S$ .

To prove this theorem, the following lemma is useful. The lemma is a version of the separating hyper-plane theorem [10] and has been derived in e.g. [11].

**LEMMA 1** *Let  $\mathcal{X}$  be a convex subset of  $\mathbb{C}^m$ , and  $F: \mathcal{X} \rightarrow \mathbb{C}^{n \times n}$  be a Hermitian-valued affine function. The following statements are equivalent.*

- (i) *The set  $\{x: x \in \mathcal{X}, F(x) < 0\}$  is empty.*
- (ii)  *$\exists$  nonzero  $H = H^* \geq 0$  s.t.  $\text{tr}(F(x)H) \geq 0 \ \forall x \in \mathcal{X}.$*

We now prove Theorem 1.

*Proof* (ii)  $\Rightarrow$  (i) is trivial. To show the converse, suppose (ii) does not hold, i.e. there is no  $S \in \mathcal{S}$  such that  $\Theta < S$ . Then, from Lemma 1, there exists a nonzero matrix  $H$  such that

$$H = H^* \geq 0, \quad \text{tr}((\Theta - S)H) \geq 0 \ \forall S \in \mathcal{S}.$$

Since  $\mathcal{S}$  is lossless, we have from property (b) of Definition 1 that

$$\text{tr}(SH) \leq 0 \ \forall S \in \mathcal{S}, \quad \text{tr}(\Theta H) \geq 0.$$

The first condition in turn implies the existence of the vectors  $\zeta_i$  in property (c), and the second condition becomes

$$\text{tr}(\Theta H) = \sum_{i=1}^r \zeta_i^* \Theta \zeta_i \geq 0.$$

Hence, there exists an index  $k$  such that  $\zeta_k^* \Theta \zeta_k \geq 0$ . Noting that  $\zeta_k \in \mathcal{G}$ , we conclude that (i) does not hold.

The significance of Theorem 1 can be explained as follows. Given a condition as in (1), Theorem 1 may be used to *equivalently* convert the condition to a numerically verifiable condition of the form given in statement (ii) of Theorem 1. To make sure that the conversion is exact, first we have to characterize the set  $\mathcal{G}$  as in (4) for some set  $\mathcal{S}$ . Then we

need to check if  $\mathcal{S}$  is lossless. Of course these steps are usually non-trivial, but can be done for some class of  $\mathcal{G}$  that is relevant to control systems analysis. We will do this next.

### 3 THE FINITE FREQUENCY KYP LEMMA

Consider the class of  $\mathcal{G}$  described by

$$\mathcal{G} := \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in \mathbb{C}^{2n} : f = j\omega g, \text{ for some } \omega \in \mathbb{R}, |\omega| \leq \omega_0 \right\}, \quad (5)$$

where  $\omega_0 > 0$  is a given real scalar. Viewing  $j\omega$  as the Laplace operator  $s$ , it is easily seen that this set is related to (input, output) signals  $(f, g)$  of an integrator. Thus it is not surprising that the set  $\mathcal{G}$  plays a key role in the analysis of dynamical systems.

The following result identifies the set  $\mathcal{S}$  that characterizes the set  $\mathcal{G}$  in (5) through the definition in (4).

**LEMMA 2** *Let a real scalar  $\omega_0$  and complex vectors  $f$  and  $g$  be given. The following statements are equivalent.*

- (i) *There exists a real scalar  $\omega$  such that  $f = j\omega g$ ,  $|\omega| \leq \omega_0$ .*
- (ii)  $\begin{bmatrix} f \\ g \end{bmatrix}^* \begin{bmatrix} Q & P \\ P & -\omega_0^2 Q \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \leq 0, \forall \text{ complex matrices } P = P^*, Q = Q^* > 0.$

*Proof* Suppose (i) holds. Then

$$\begin{bmatrix} f \\ g \end{bmatrix}^* \begin{bmatrix} Q & P \\ P & -\omega_0^2 Q \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = (\omega^2 - \omega_0^2)(g^* Q g) \leq 0$$

and hence (ii) holds. Conversely, if (ii) is satisfied,

$$\text{tr}(ff^* - \omega_0^2 gg^*)Q + \text{tr}(gf^* + fg^*)P \leq 0$$

holds for all  $P = P^*$  and  $Q = Q^* > 0$ . It can readily be verified that this implies

$$ff^* - \omega_0^2 gg^* \leq 0, \quad gf^* + fg^* = 0.$$

It now follows from Lemma III.4 of [11] that (i) holds.

Let us now give a result that shows the losslessness of the set  $\mathcal{S}$  related to  $\mathcal{G}$  defined in (5). Its proof is rather technical and will be given later to keep the presentation streamlined.

**LEMMA 3** *Let a scalar  $\omega_0 > 0$  and a matrix  $F \in \mathbb{C}^{2n \times k}$  be given. Define a subset of Hermitian matrices by*

$$\mathcal{S} := \left\{ F^* \begin{bmatrix} Q & P \\ P & -\omega_0^2 Q \end{bmatrix} F : P = P^*, Q = Q^* > 0 \right\}.$$

*Then the set  $\mathcal{S}$  is lossless.*

The following theorem is a generalization of the KYP lemma [6,7] where a frequency domain condition is required to hold only for a given low frequency band. The result is a simple consequence of the generalized S-procedure.

**THEOREM 2** *Let a scalar  $\omega_0 > 0$  and matrices  $A \in \mathbb{C}^{n \times n}$ ,  $B \in \mathbb{C}^{n \times m}$  and a Hermitian matrix  $\Theta \in \mathbb{C}^{(n+m) \times (n+m)}$  be given. Suppose  $A$  has no eigenvalues on the imaginary axis. Then the following statements are equivalent.*

(i) *The finite frequency condition*

$$\begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix} < 0, \quad \forall |\omega| \leq \omega_0$$

*holds.*

(ii) *There exist Hermitian matrices  $P, Q \in \mathbb{C}^{n \times n}$  such that  $Q > 0$  and*

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -Q & P \\ P & \omega_0^2 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta < 0.$$

*If matrices  $A, B$  and  $\Theta$  are all real, the equivalence still holds when restricting  $P$  and  $Q$  to be real.*

*Proof* Note that (i) holds if and only if

$$\zeta^* \Theta \zeta < 0 \quad \forall \zeta \in \mathcal{G}$$

where

$$\mathcal{G} := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathbb{C}^{n+m} : w \neq 0, j\omega x = Ax + Bw \text{ for some } \omega \in \mathbb{R}, |\omega| \leq \omega_0 \right\}.$$

Defining

$$\begin{bmatrix} f \\ g \end{bmatrix} := F \begin{bmatrix} x \\ w \end{bmatrix}, \quad F := \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}$$

and applying Lemma 2, the set  $\mathcal{G}$  can be characterized as

$$\mathcal{G} = \{\zeta \neq 0 : \zeta^* S \zeta \leq 0 \ \forall S \in \mathcal{S}\}$$

where

$$\mathcal{S} := \left\{ F^* \begin{bmatrix} Q & P \\ P & -\omega_0^2 Q \end{bmatrix} F : P = P^*, Q = Q^* > 0 \right\}.$$

From Lemma 3, the set  $\mathcal{S}$  is lossless and hence the  $S$ -procedure in Theorem 1 yields (i)  $\Leftrightarrow$  (ii).

Finally, to prove the real case result, assume that there exist (complex) Hermitian matrices  $P$  and  $Q$  satisfying the condition in statement (ii). Then, noting that

$$(M + jN) = (M + jN)^* > 0 \Leftrightarrow \begin{bmatrix} M & -N \\ N & M \end{bmatrix} = \begin{bmatrix} M & -N \\ N & M \end{bmatrix}' > 0 \quad (6)$$

holds for any real square matrices  $M$  and  $N$ , one can show that the real parts of  $P$  and  $Q$  also satisfy the same condition.

A simple change of variables in Theorem 2 yields a characterization of another frequency domain condition where the inequality is required to hold in an arbitrarily given frequency interval.

**COROLLARY 1** *Let real scalars  $\omega_1 \leq \omega_2$ , matrices  $A \in \mathbb{C}^{n \times n}$ ,  $B \in \mathbb{C}^{n \times m}$  and a Hermitian matrix  $\Theta \in \mathbb{C}^{(n+m) \times (n+m)}$  be given. Suppose  $A$  has no eigenvalues on the imaginary axis. Then the following statements*

are equivalent.

(i) *The finite frequency condition*

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} < 0, \quad \forall \omega_1 \leq \omega \leq \omega_2 \quad (7)$$

holds.

(ii) *There exist Hermitian matrices  $P, Q \in \mathbb{C}^{n \times n}$  such that  $Q > 0$  and*

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -Q & P + j\omega_c Q \\ P - j\omega_c Q & -\omega_1 \omega_2 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta < 0, \quad (8)$$

where  $\omega_c := (\omega_1 + \omega_2)/2$ .

*Proof* Note that  $\omega_1 \leq \omega \leq \omega_2$  is equivalent to  $|\hat{\omega}| \leq \hat{\omega}_{\max}$  where

$$\hat{\omega} = \omega - \omega_c, \quad \hat{\omega}_{\max} = (\omega_2 - \omega_1)/2.$$

Hence, the result follows by applying Theorem 2 to  $(\hat{A}, B, \Theta)$  with  $\hat{\omega}$  via the following transformation:

$$j\omega I - A = j\hat{\omega} I - \hat{A}, \quad \hat{A} := A - j\omega_c I.$$

When  $A, B$  and  $\Theta$  are real matrices, one can show the following: If inequality (8) holds for

$$\begin{aligned} \omega_1 &:= \alpha, \quad \omega_2 := \beta, \\ P &:= P_R + jP_I, \quad Q := Q_R + jQ_I > 0 \end{aligned}$$

then the same inequality holds for

$$\begin{aligned} \omega_1 &:= -\beta, \quad \omega_2 := -\alpha, \\ P &:= P_R - jP_I, \quad Q = Q_R - jQ_I > 0. \end{aligned}$$

Thus the frequency domain condition (7) holds for  $\omega_1 \leq \omega \leq \omega_2$ , if and only if the same condition holds for  $-\omega_2 \leq \omega \leq -\omega_1$ .

When  $A$  and  $B$  are real, the finite frequency condition in Corollary 1 can be characterized by an LMI involving real matrices only. Such



characterization is directly useful for numerical computation. The result follows from a straightforward application of the identity (6) and hence the proof is omitted.

**COROLLARY 2** *Consider the finite frequency condition in Corollary 1. If  $A$  and  $B$  are real matrices, the condition is equivalent to the following:*

(iii) *There exist real symmetric matrices  $\mathcal{P}, \mathcal{Q} \in \mathbb{R}^{2n \times 2n}$  of the form*

$$\mathcal{P} = \begin{bmatrix} P_R & -P_I \\ P_I & P_R \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} Q_R & -Q_I \\ Q_I & Q_R \end{bmatrix},$$

*satisfying  $\mathcal{Q} > 0$  and*

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}' \begin{bmatrix} -\mathcal{Q} & \mathcal{P} + J\omega_c \mathcal{Q} \\ \mathcal{P} - J\omega_c \mathcal{Q} & -\omega_1 \omega_2 \mathcal{Q} \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Phi < 0,$$

*where*

$$J := \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \quad A := \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad B := \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$$

*and  $\Phi$  is defined in terms of the real and the imaginary parts of  $\Theta$  as follows:*

$$\begin{aligned} \Theta &= \begin{bmatrix} U_R & V_R \\ V_R' & W_R \end{bmatrix} + j \begin{bmatrix} U_I & V_I \\ -V_I' & W_I \end{bmatrix}, \\ \Phi &:= \begin{bmatrix} U & V \\ V' & W \end{bmatrix}, \quad U := \begin{bmatrix} U_R & -U_I \\ U_I & U_R \end{bmatrix}, \\ V &:= \begin{bmatrix} V_R & -V_I \\ V_I & V_R \end{bmatrix}, \quad W := \begin{bmatrix} W_R & -W_I \\ W_I & W_R \end{bmatrix}. \end{aligned}$$

#### 4 CONNECTION TO THE $(D, G)$ -SCALING

The finite frequency KYP lemma (Theorem 2) shown in the previous section can also be derived through the losslessness theorem of the  $(D, G)$ -scaling upper bound of mixed  $\mu$  [11]. In that case, we need some

restrictions on matrix  $\Theta$  to allow for an appropriate loop-shifting and its proof will no longer be self-contained, for the necessity proof relies on the losslessness of the  $(D, G)$ -scaling shown in [11]. Nevertheless, it would be of interest to outline the derivation of the finite frequency KYP lemma through the  $(D, G)$ -scaling.

Let us first derive the finite frequency bounded-real lemma which is a special case of the finite frequency KYP lemma. Consider the  $m \times p$  transfer function matrix

$$G(s) := C(sI - A)^{-1}B + D,$$

where matrices  $A, B, C$  and  $D$  are possibly complex. Suppose  $A$  has no eigenvalues on the imaginary axis. Then it can readily be verified [9] that the following identity holds for all real scalars  $\omega$  and  $\omega_0 > 0$ :

$$G(j\omega) = C(I - \delta A)^{-1} \delta B + D =: G(\delta),$$

where

$$\delta := \omega/\omega_0, \\ M := \begin{bmatrix} A & B \\ C & D \end{bmatrix} := \begin{bmatrix} j\omega_0 A^{-1} & A^{-1}B \\ -j\omega_0 CA^{-1} & D - CA^{-1}B \end{bmatrix}.$$

From the standard  $\mu$ -analysis, we have

$$\begin{aligned} \|G(j\omega)\| < 1, \quad \forall |\omega| \leq \omega_0 &\Leftrightarrow \|G(\delta)\| < 1, \quad \forall |\delta| \leq 1 \\ &\Leftrightarrow \det(I - M\nabla) \neq 0, \quad \forall \nabla \in \nabla, \end{aligned}$$

where

$$\nabla := \{\text{diag}(\delta I, \Delta): \delta \in \mathbb{R}, \Delta \in \mathbb{C}^{p \times m}, |\delta| \leq 1, \|\Delta\| \leq 1\}.$$

Using the losslessness of the  $(D, G)$ -scaling with respect to the uncertainty  $\nabla$  consisting of one repeated real scalar  $\delta$  and one full-block complex matrix  $\Delta$  [11], the last condition is equivalent to the existence of complex matrices  $\mathcal{D} = \mathcal{D}^* > 0$  and  $\mathcal{G} = -\mathcal{G}^*$  such that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} \mathcal{D} & \mathcal{G} \\ \mathcal{G}^* & -\mathcal{D} \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0.$$

Now, defining

$$P := j\mathcal{G}^*/\omega_0, \quad Q := \mathcal{D}/\omega_0^2$$

the congruent transformation by  $\begin{bmatrix} A & B \\ 0 & -j\omega_0 I \end{bmatrix}$  yields

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^* \begin{bmatrix} -Q & P \\ P & \omega_0^2 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0.$$

Clearly,  $P = P^*$  and  $Q = Q^* > 0$ . Thus the existence of such  $P$  and  $Q$  is necessary and sufficient for the finite frequency bounded-real condition to hold.

We now consider the condition

$$\begin{bmatrix} (j\omega I - A)^{-1} B \\ I_p \end{bmatrix}^* \Theta \begin{bmatrix} (j\omega I - A)^{-1} B \\ I_p \end{bmatrix} < 0, \quad \forall |\omega| \leq \omega_0. \quad (9)$$

Clearly,  $\Theta$  must have at least  $p$  negative eigenvalues in order for this condition to hold. On the other hand, if all the eigenvalues are negative, the condition becomes trivial. Hence, it is reasonable to assume that  $\Theta$  has both positive and negative eigenvalues, in which case, it can be written as

$$\Theta = \begin{bmatrix} C_1 & D_1 \\ C_2 & D_2 \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} C_1 & D_1 \\ C_2 & D_2 \end{bmatrix}.$$

Let us also assume that  $D_2$  is square ( $p \times p$ ) and nonsingular. This is a restrictive assumption. Using the above expression for  $\Theta$ , the condition in (9) can be described by

$$G_1(j\omega)^* G_1(j\omega) < G_2(j\omega)^* G_2(j\omega), \quad \forall |\omega| \leq \omega_0,$$

where

$$G_i(s) := C_i(sI - A)^{-1} B + D_i \quad (i = 1, 2).$$

This condition is in turn equivalent to

$$\|G(j\omega)\| < 1, \quad \forall |\omega| \leq \omega_0, \quad G(s) := G_1(s)G_2(s)^{-1}.$$

It can be verified that a state space realization for  $G(s)$  is given by

$$G(s) = \left( \begin{array}{c|c} A - BD_2^{-1}C_2 & BD_2^{-1} \\ \hline C_1 - D_1D_2^{-1}C_2 & D_1D_2^{-1} \end{array} \right).$$

Applying the finite frequency bounded-real condition to  $G(s)$  and performing the congruent transformation with  $\begin{bmatrix} I & 0 \\ C_2 & D_2 \end{bmatrix}$ , it can be shown that the finite frequency KYP lemma (Theorem 2) holds.

## 5 PROOF OF THE LOSSLESSNESS OF THE SET $\mathcal{S}$

In this section, we prove Lemma 3. The following two lemmas are instrumental for the proof. Below,  $(\cdot)^\dagger$  denotes the Moore–Penrose inverse of a matrix.

**LEMMA 4** *Let complex matrices  $R$  and  $S$  be given. Suppose*

$$\| \begin{bmatrix} R & S \end{bmatrix} \| \leq 1, \quad R + R^* = 0. \quad (10)$$

*Then there exists a matrix  $Q$  such that*

$$\left\| \begin{bmatrix} R & S \\ -S^* & Q \end{bmatrix} \right\| \leq 1, \quad Q + Q^* = 0. \quad (11)$$

*Moreover, one such  $Q$  is given by*

$$Q = -S^*R(I + R^2)^\dagger S.$$

*Proof* From the supposition, we have  $\|R\| \leq 1$  and hence  $I - RR^* \geq 0$ . Let  $\Omega := (I - RR^*)^{1/2}$ . From (10),

$$RR^* + SS^* \leq I \Rightarrow SS^* \leq \Omega^2.$$

This implies (e.g. [12]) that there exists a matrix  $C$  such that

$$S = \Omega C, \quad \|C\| \leq 1.$$

Let

$$Q := -S^* \Omega^\dagger R \Omega^\dagger S = -S^* R (I - R R^*)^\dagger S.$$

Clearly,  $Q$  is skew Hermitian. Note that

$$\begin{aligned} \left\| \begin{bmatrix} R & S \\ -S^* & Q \end{bmatrix} \right\| &= \left\| \begin{bmatrix} R & \Omega C \\ -C^* \Omega & C^* \hat{Q} C \end{bmatrix} \right\| \\ &\leq \left\| \begin{bmatrix} I & 0 \\ 0 & -C^* \end{bmatrix} \right\| \left\| \begin{bmatrix} R & \Omega \\ -\Omega & \hat{Q} \end{bmatrix} \right\| \left\| \begin{bmatrix} I & 0 \\ 0 & -C \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} R & \Omega \\ -\Omega & \hat{Q} \end{bmatrix} \right\|, \end{aligned}$$

where  $\hat{Q} := -\Omega \Omega^\dagger R \Omega^\dagger \Omega$  and the last inequality holds due to  $\|C\| \leq 1$ . It can be verified that

$$R \Omega + \Omega R^* = 0.$$

Repeated use of this identity, after some manipulations, yields

$$\begin{bmatrix} R & \Omega \\ -\Omega & \hat{Q} \end{bmatrix} \begin{bmatrix} R & \Omega \\ -\Omega & \hat{Q} \end{bmatrix}^* = \begin{bmatrix} I & 0 \\ 0 & I - R(I - \Omega \Omega^\dagger) R^* \end{bmatrix} \leq I,$$

where the last inequality is due to the following fact:

$$I - \Omega \Omega^\dagger \geq 0 \Rightarrow 0 \leq I - R(I - \Omega \Omega^\dagger) R^* \leq I.$$

Hence we conclude that the norm condition in (11) holds.

**LEMMA 5** *Let complex matrices  $Z$  and  $W$  of the same dimensions be given. The following statements are equivalent.*

- (i)  $WW^* \leq ZZ^*$  and  $ZW^* + WZ^* = 0$ .
- (ii) *There exists a complex matrix  $\Delta$  such that*

$$W = Z\Delta, \quad \|\Delta\| \leq 1, \quad \Delta + \Delta^* = 0.$$

*Proof* (ii)  $\Rightarrow$  (i) is trivial. To show the converse, suppose (i) holds. Then there exists  $\nabla$  such that

$$W = Z\nabla, \quad \|\nabla\| \leq 1.$$

This  $\nabla$  satisfies

$$Z(\nabla + \nabla^*)Z^* = 0.$$

If  $Z^*Z > 0$ , then  $\nabla + \nabla^* = 0$  and we are done. So consider the case  $Z^*Z \not> 0$ . Let  $V$  be a Unitary matrix such that

$$ZV = [Z_1 \ 0],$$

where  $Z_1$  is full column rank. Define  $R$  and  $S$  by

$$\begin{bmatrix} R & S \\ * & * \end{bmatrix} := V^* \nabla V,$$

where  $R$  is square with its dimension equal to the rank of  $Z$  and  $*$  denotes irrelevant entries. Then

$$\begin{aligned} \|[R \ S]\| &\leq 1 \Leftrightarrow \|\nabla\| \leq 1 \\ R + R^* &= 0 \Leftrightarrow Z(\nabla + \nabla^*)Z^* = Z_1(R + R^*)Z_1^* = 0. \end{aligned}$$

From Lemma 4, there exists  $Q$  such that

$$\Delta := V \begin{bmatrix} R & S \\ -S^* & Q \end{bmatrix} V^*, \quad \|\Delta\| \leq 1, \quad \Delta + \Delta^* = 0.$$

For this  $\Delta$ , we have

$$Z\Delta = [Z_1 \ 0] \begin{bmatrix} R & S \\ -S^* & Q \end{bmatrix} V^* = Z\nabla = W.$$

Hence we conclude that (i)  $\Rightarrow$  (ii).

We are now ready to prove Lemma 3.

*Proof* Properties (a) and (b) in Definition 1 are easily verified. To show property (c), let  $H$  be a nonzero matrix such that

$$H = H^* \geq 0, \quad \text{tr}(HS) \leq 0 \quad \forall S \in \mathcal{S}. \quad (12)$$

Since  $H$  is positive semi-definite, it admits a full rank factor  $H = GG^*$ ,  $G \in \mathbb{C}^{k \times r}$  where  $r$  is the rank of  $H$ . Defining

$$\begin{bmatrix} W \\ Z \end{bmatrix} := FG, \quad W, Z \in \mathbb{C}^{n \times r},$$

the latter condition in (12) can be written

$$\begin{aligned} \operatorname{tr}(WW^* - \omega_0^2 ZZ^*)Q + \operatorname{tr}(WZ^* + ZW^*)P &\leq 0 \\ \forall P = P^*, Q = Q^* &> 0. \end{aligned}$$

It can readily be verified that this condition is equivalent to

$$WW^* \leq \omega_0^2 ZZ^*, \quad WZ^* + ZW^* = 0.$$

From Lemma 5, there exists a matrix  $\Delta \in \mathbb{C}^{r \times r}$  such that

$$W = \omega_0 Z \Delta, \quad \|\Delta\| \leq 1, \quad \Delta + \Delta^* = 0.$$

Since  $\Delta$  is skew-Hermitian with norm less than or equal to one, its spectral decomposition yields

$$\Delta = \sum_{i=1}^r \lambda_i u_i u_i^*, \quad |\lambda_i| \leq 1, \quad \lambda_i + \bar{\lambda}_i = 0, \quad \sum_{i=1}^r u_i u_i^* = I.$$

For  $i = 1, \dots, r$ , define

$$\zeta_i := G u_i, \quad \begin{bmatrix} w_i \\ z_i \end{bmatrix} := \begin{bmatrix} W \\ Z \end{bmatrix} u_i = F \zeta_i, \quad w_i, z_i \in \mathbb{C}^n.$$

Then  $H = \sum_{i=1}^r \zeta_i \zeta_i^*$  and

$$W u_i = \omega_0 Z \Delta u_i \Rightarrow w_i = \lambda_i \omega_0 z_i.$$

Hence we have

$$\begin{aligned} w_i w_i^* &= \omega_0^2 |\lambda_i|^2 z_i z_i^* \leq \omega_0^2 z_i z_i^*, \\ w_i z_i^* + z_i w_i^* &= \omega_0 (\lambda_i + \bar{\lambda}_i) z_i z_i^* = 0. \end{aligned}$$

These conditions imply

$$\operatorname{tr}(\zeta_i \zeta_i^* S) = \zeta_i^* S \zeta_i \leq 0 \quad \forall S \in \mathcal{S}$$

and we conclude that  $\mathcal{S}$  satisfies property (c) of Definition 1.

## 6 CONCLUSION

We have given a generalization of the  $S$ -procedure, a powerful tool in control and optimization theories. As an application of the generalized  $S$ -procedure, the finite frequency KYP lemma is derived. These results are expected to be useful for control systems analysis and synthesis.

## Acknowledgments

The authors gratefully acknowledge helpful discussions with L. El Ghaoui and S. Hara.

## References

- [1] V.A. Yakubovič.  $S$ -procedure in nonlinear control theory. *Vestnik Leningrad Univ.*, **1**: 62–77, 1971.
- [2] A. Megretsky and S. Treil. Power distribution inequalities in optimization and robustness of uncertain systems. *J. Math. Syst. Estim. Contr.*, **3**(3): 301–319, 1993.
- [3] S.P. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM Studies in Applied Mathematics, 1994.
- [4] V.A. Yakubovich. Nonconvex optimization problem: the infinite-horizon linear-quadratic control problem with quadratic constraints. *Sys. Contr. Lett.*, **19**: 13–22, 1992.
- [5] V.A. Yakubovič. Minimization of quadratic functionals under quadratic constraints and the necessity of a frequency condition in the quadratic criterion for absolute stability of nonlinear control systems. *Soviet Math. Dokl.*, **14**(2): 593–597, 1973.
- [6] B.D.O. Anderson. A system theory criterion for positive real matrices. *SIAM J. Contr.*, **5**: 171–182, 1967.
- [7] A. Rantzer. On the Kalman–Yakubovich–Popov lemma. *Sys. Contr. Lett.*, **28**(1), 1996.
- [8] T. Iwasaki. Integrated system design by separation. *Proc. IEEE Conf. Contr. Appl.*, 97–102, August 1999.
- [9] A. Sideris. Elimination of frequency search from robustness tests. *IEEE Trans. Auto. Contr.*, **37**(10): 1635–1640, 1992.
- [10] D.G. Luenberger. *Optimization by Vector Space Methods*. John Wiley, 1968.
- [11] G. Meinsma, Y. Shrivastava and M. Fu. A dual formulation of mixed  $\mu$  and on the losslessness of  $(D, G)$  scaling. *IEEE Trans. Auto. Contr.*, **42**(7): 1032–1036, 1997.
- [12] R.E. Skelton, T. Iwasaki and K.M. Grigoriadis. *A Unified Algebraic Approach to Linear Control Design*. Taylor & Francis, 1997.



## Special Issue on Intelligent Computational Methods for Financial Engineering

### Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods:** artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning

- **Application fields:** asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects:** decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site <http://www.hindawi.com/journals/jamds/>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/>, according to the following timetable:

|                        |                  |
|------------------------|------------------|
| Manuscript Due         | December 1, 2008 |
| First Round of Reviews | March 1, 2009    |
| Publication Date       | June 1, 2009     |

### Guest Editors

**Lean Yu**, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; [yulean@amss.ac.cn](mailto:yulean@amss.ac.cn)

**Shouyang Wang**, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; [sywang@amss.ac.cn](mailto:sywang@amss.ac.cn)

**K. K. Lai**, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; [mskkklai@cityu.edu.hk](mailto:mskkklai@cityu.edu.hk)

## Special Issue on Short Range Phenomena: Modeling, Computational Aspects, and Applications

### Call for Papers

In recent years, the mathematical formalism of impulsive systems (based on impulsive differential equations) has tried to join together the rigorous aspects from continuous systems formalism and the wide range of applications of discrete systems formalism. They were introduced to handle many evolution processes which are subject to singular short-term perturbations. Abrupt changes must be approached with mathematical and technical aspects dealing with the final evolution of such impulsive sources, whose effects are entirely transferred to the new state of the systems like transitions in quantum mechanics. Modern aspects in physics (quantum theory) and mathematics (wavelets, fractal theory) should be expedient in modeling short range phenomena and describing dynamics of perturbations and transitions in natural systems (advanced materials science) and advanced systems (optic, electronic, and quantum devices).

Thus, a special issue on all theoretical, computational, and practical aspects of modeling short range phenomena would be an opportunity of extending the research field of wavelets analysis, fractal theory, and applied mathematics (signal processing, control theory) for presenting new fundamental aspects in science and engineering. We are soliciting original high-quality research papers on topics of interest connected with modeling short range phenomena that include but are not limited to the following main topics:

- Mathematical aspects of pulse generation
- Dynamical and computational aspects of pulse measurement
- Wavelets analysis of localized space-time phenomena
- Stochastic aspects of pulses, sequences of pulses and time series

Authors should follow the Mathematical Problems in Engineering manuscript format described at the journal site <http://www.hindawi.com/journals/mpe/>. Prospective authors should submit an electronic copy of their complete manuscript through the journal's Manuscript Tracking Sys-

tem at <http://mts.hindawi.com/>, according to the following timetable:

|                        |                   |
|------------------------|-------------------|
| Manuscript Due         | June 1, 2008      |
| First Round of Reviews | September 1, 2008 |
| Publication Date       | December 1, 2008  |

### Guest Editors

**Carlo Cattani**, DiFarma, University of Salerno, Via Ponte Don Melillo, 84084 Fisciano (SA), Italy; [ccattani@unisa.it](mailto:ccattani@unisa.it)

**Ming Li**, Department of Electronic Science & Technology, School of Information Science & Technology, East China Normal University, 3663 Zhongshan Bei Road, Shanghai 200062, China; [mli@ee.ecnu.edu.cn](mailto:mli@ee.ecnu.edu.cn)

**Cristian Toma**, Faculty of Applied Sciences, Politehnica University, Hagi-Ghita 81, Bucharest Street 060032, Romania; [cgtoma@physics.pub.ro](mailto:cgtoma@physics.pub.ro)