This thesis contains no material which has been accepted for the award of any other degree or diploma in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. I give consent to this copy of my thesis, when deposited in the University Library, being made available for loan and photocopying subject to the provisions of the Copyright Act 1968.

Miss Rebecca Smith BMath(Hons)
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This thesis aims to provide an overview of the current works in the field of fractal sets produced via iterative mapping systems on circles. Chapter 1 will provide a review of the relevant preliminary material.

Chapter 2 begins by reviewing the work of Frame, Mandelbrot and Neger (2005), which details the use of circle inversion iteration in the creation of fractal limit sets in the extended real plane $\mathbb{R}^2$, where $\mathbb{R}^2 = \mathbb{R}^2 \cup \infty$. The work of Frame et al (2005) primarily focussed on non-overlapping circles in the plane, but they also discuss the ramification points of contact have on the resulting iterative inversion limit set. We extend the work of Frame et al (2005) by developing a classification theorem for $n$ non-overlapping circles in $\mathbb{R}^2$. This theorem predicts the structure of the limit set based on the placement of the initial inversion circles.

In Chapter 3 the circle inversion mappings will be applied to overlapping circles as well as being extended into three dimensions in the form of spherical inversion.

Chapter 4 focuses on iteration in the extended complex plane $\hat{\mathbb{C}}$, where $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$. Within $\hat{\mathbb{C}}$ iteration of circle pairing Möbius transformations will be studied. Fractal sets produced as the limit set of a system of circle pairing Möbius transformations has been extensively analysed by Mumford, Series and Wright (2002) in the text *Indra's Pearls: The Vision of Felix Klein*. A review of this text will show the various fractal limit sets produced by such iterative systems and how careful selection of the circle pairing transformations produce fractal limit sets.

In Chapter 5 we will study the limit sets of iterative systems of mappings that are variations of the circle inversion mappings previously discussed in Chapter 2. These mappings, applied to sets contained within $\hat{\mathbb{C}}$, incorporate reflection within the circle of inversion. These Möbius transformations will be termed complex inversion mappings. Systems of these complex inversion mappings incorporating reflection will be shown to produce fractal limit sets.

Although extensive work has already been done in the field of fractal geometry and iterative mappings on circles, the field is ever expanding with many possible avenues for expansion. In Chapter 6 we will outline some further avenues for development.
Chapter 1 Preliminaries

§1 Fractals

Benoit Mandelbrot coined the term fractal in 1975 (Shiode, 1998). This word comes from the Latin term ‘fractus’, which means broken. Falconer (2003) describes fractal sets as sets which are

“Too irregular to be effectively studied in the confines of traditional and classical geometry.”

At present, there does not exist a universally agreed upon definition for ‘fractal’; Barnsley (1988) defines fractal as

“A fractal is a geometrically complex subset of a geometrically simple space”.

Whilst this definition conveys the idea of a fractal, it is not precise in that it does not specify what is meant by ‘geometrically complex’ or ‘geometrically simple’. Thus, Barnsley’s definition is usually accompanied by examples such as The Sierpinski Triangle, The Koch Curve and The Cantor Middle Third Set.

Falconer (2003) suggests that a more practical approach to the problem of defining what is meant by ‘fractal’ is to study the characteristics of the set and from these characteristics classify it as either fractal or non-fractal. To be a fractal the set \( F \) would usually conform to most, if not all, of the following criteria:

1. \( F \) has a fine structure, i.e. detail on arbitrarily small scales.
2. \( F \) is too irregular to be described using traditional geometrical language, both locally and globally.
3. Often \( F \) has some form of self-similarity, perhaps statistical.
4. Usually, the fractal dimension of \( F \) (defined in some way) is greater than its topological dimension.
5. In most cases of interest \( F \) is defined in a very simple way, perhaps recursively.
The Sierpinski Triangle, The Koch Curve and The Cantor Middle Third Set obey all of the above criteria. Some sets which do not satisfy all of the criterion points may still be fractal. Some sets once considered non-fractal have been reclassified after comprehensive study. The ‘father’ of fractal geometry, Benoit Mandelbrot, produced one of the most famous of all fractals in 1979; this fractal is called The Mandelbrot Set (Peitgen et al, 1993). The Mandelbrot Set can be defined as the set of those complex values \(c\) for which the orbit of zero under iteration of the complex polynomial

\[ z_{n+1} = z_n^2 + c \]

remains bounded (Falconer, 2003 and Peitgen et al, 1992). It can be generated graphically, as seen in figure 1.101. The Mandelbrot Set is classified fractal, yet it was initially not considered fractal as it does not display exact self-similarity. Mandelbrot (Peitgen et al, 1992) states:

“This set is not truly fractal by most definitions; we may call it a borderline fractal, a limit fractal that contains many fractals. Compared to actual fractals, its structures are more numerous, its harmonies are richer, and its unexpectedness is more unexpected”.

Figure 1.101: The Mandelbrot Set (Wikipedia, 2009)

The ideas of self-similarity and fractal dimension, concepts required for criterion points three and four, were developed as a means by which to analyse and compare complex sets. Barnsley (1988) describes fractal dimension as an:
“Attempt to quantify a subjective feeling which we have about how densely the fractal occupies the metric space in which it lies”.

Intuitively then, we can think of dimension as a measure of the ‘complexity’ of a set.

Fractal geometry, as a mathematical field of study, is relatively new. However it has found application in many branches of science. Natural structures are sometimes inaccurately referred to as being fractal. The problem with ‘fractals from nature’ is that they do not possess detail on all scales. These structures can still be analysed, using the tools of fractal geometry.

§2 Convergent Sequences and Limit Points

The concept of convergent sequences and limit points form the basis for the analysis of fractals constructed under an iterative system.

**Definition 1.201 - Convergence** (Barnsley, 1988)

A sequence \( \{x_n\}_{n=1}^{\infty} \) of points in a metric space \((X, d)\) is said to converge to a point \(x \in X\) if, for a given number \(\varepsilon > 0\), there is an integer \(N > 0\) so that

\[
d(x_n, x) < \varepsilon \quad \text{for all } n > N.
\]

In this case the point \(x \in X\), to which the sequence converges, is called the **limit** of the sequence, and we use the notation

\[
\lim_{n \to \infty} x_n = x.
\]

When studying iterative systems we will primarily focus on contraction mappings.
Definition 1.202 (Barnsley, 1988)

A map $w$ from the metric space $(X, d)$ to itself is called **contractive** or a **contraction mapping** if there exists a constant $0 \leq s < 1$ such that

$$d(w(x), w(y)) \leq sd(x, y) \text{ for all } x, y \in X.$$ 

Any such number $s$ is called a contractivity factor for $w$.

The minimum value $s$ such that Definition 1.202 is true is called the Lipschitz constant of $w$ and is defined

$$\text{Lip } w = \sup_{x,y \in X} \frac{d(w(x), w(y))}{d(x, y)}.$$ 

A property of contraction mappings that proves helpful in the study of fractal sets is the existence of a unique fixed point (**limit point**) for the map under iteration.

**Theorem 1.201 - Contraction Mapping Principle** (Hart, 1996)

Let $(X, d)$ be a complete metric space and let $w : X \rightarrow X$ be a contraction on $X$. Then $w$ has a unique fixed point $x_0$ in $X$.

**Proof:**

See Appendices

**Corollary 1.201** (Hart, 1996)

Let $w, X$ be as in Theorem 1.201. Then all points in $X$ converge to the fixed point under iteration of $w$.

**Proof:**

See Appendices
§3 Iterated Function Systems (IFSs)

In 1988 Barnsley popularised, in his book *Fractals Everywhere*, the idea of fractal construction by means of an Iterated Function System (IFS). Given a metric space \((X, d)\) where \(X\) is complete and \(d\) is a metric defined on \(X\) an IFS is a set of contraction mappings defined on \(X\). Starting with any given subset of \(X\), iteration under these maps will converge to the same set. This set, the limit set of the IFS, is termed the attractor for the IFS. Barnsley’s Fern, named after Michael Barnsley, is one such attractor, as shown in figure 1.301.

![Barnsley’s Fern](image)

Figure 1.301 - Barnsley’s Fern (Riddle, 2009)

More formally an IFS can be defined as follows:

**Definition 1.301 - Iteration Function System (IFS)** (Barnsley, 1988)

An Iterated Function System consists of a complete metric space \((X, d)\) together with a finite set of contraction mappings \(w_n : X \to X\), with respective contractivity factor \(s_n\), for \(n = 1, 2, ..., N\). The notation for the IFS is \(\{X : w_n, n = 1, 2, ..., N\}\) and its contractivity factor is \(s = \text{Max}\{s_n : n = 1, 2, ..., N\}\).

In practice, IFSs frequently use only affine mappings in their construction. Barnsley’s definition of an IFS does not specify affine mappings yet Barnsley uses only mappings of an affine nature in his work. Given the somewhat generic nature of Barnsley’s definition mathematicians such as Hart (1996) choose to define IFSs with the property that only contractive affine maps can be used.
Definition 1.302 - Iteration Function System (IFS) (Hart, 1996)

An Iterated Function System is a finite collection \( w = \{ w_i \}_{i=1}^N \) of affine contractions \( w_i : X \rightarrow X \) defined on \( (X, d) \).

In any complete metric space the operation of an IFS can be summarised by The Fundamental Theorem of Iterated Function Systems. A proof of this theorem may be found in Barnsley (1988) in which the following notation is used. Given a complete metric space \( (X, d) \), \( (\mathcal{H}(X), h(d)) \) denotes the corresponding space of non-empty compact subsets with Hausdorff metric \( h(d) \).

Definition 1.303 - Hausdorff Metric \((h(d))\)(Hart, 1996)

The Hausdorff metric \( h \) measures the distance between sets \( A, B \subseteq X \) as

\[
h(A, B) = \sup_{x \in A, y \in B} \{d(x, B), d(y, A)\}.
\]

Informally, let \( x \in A \) be the point furthest from any point in \( B \) and \( y \in B \) be furthest from any point in \( A \). The larger of these two maximal distances determines the Hausdorff distance between sets \( A \) and \( B \).

Theorem 1.301 - Fundamental Theorem of Iterated Function Systems (Barnsley, 1988)

Let \( \{X : w_n, n = 1, 2, \ldots, N\} \) be an Iterated Function System with contractivity factor \( s \).

Then the transformation \( W : \mathcal{H}(X) \rightarrow \mathcal{H}(X) \) defined by

\[
W(B) = \bigcup_{n=1}^{N} w_n(B)
\]

for \( B \in \mathcal{H}(X) \), is a contraction mapping on the complete metric space \( (\mathcal{H}(X), h(d)) \) with contractivity factor \( s \). Then the unique fixed point of \( W, \ A \in \mathcal{H}(X) \), obeys
\[ A = W(A) = \bigcup_{n=1}^{N} w_n(A) \]

and is given by \( A = \lim_{n \to \infty} W^n(B) \) for any \( B \subseteq \mathcal{X} \), where \( W^n(B) \) denotes the \( n^{th} \) fold composition of the transformation \( W \) in relation to the point \( B \). \( W \) is termed the Hutchinson operator.

Basically, The Fundamental Theorem of Iteration Function Systems states that the limiting set of an IFS is fixed under the IFS. This set is called the attractor of the IFS.

**Example:** (Barnsley, 1988 and Weisstein, 2009)

Barnsley’s Fern is constructed within the complete metric space \( \hat{\mathbf{R}}^2 \), where \( \hat{\mathbf{R}}^n = \mathbf{R}^n \cup \infty \) is the extended \( n^{th} \) dimensional real metric space, using the standard Euclidian metric. The contraction mappings \( \{ w_i : i = 1,2,3,4 \} \) associated with Barnsley’s Fern are as follows.

\[
\begin{align*}
\{ w_i : i = 1,2,3,4 \} & : \hat{\mathbf{R}}^2 \to \hat{\mathbf{R}}^2 \\

w_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 0.16 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{(Red)} \\
w_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0.85 & 0.04 \\ -0.4 & 0.85 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0.16 \end{pmatrix} \quad \text{(Blue)} \\
w_3 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 0.2 & -0.26 \\ 0.23 & 0.22 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0.16 \end{pmatrix} \quad \text{(Green)} \\
w_4 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0.44 \end{pmatrix} \quad \text{(Purple)}
\end{align*}
\]

Barnsley’s Fern is the limiting set for this IFS, shown in figure 1.302.
Figure 1.303 shows the effect that the above mentioned mappings have on the initial set. Under iteration of \( w_1 \) the resulting images ‘flatten out’ to form the stem. Iteration of \( w_2 \) constructs the overall form of the fern and iteration of \( w_3 \) and \( w_4 \) form the lower fronds of the fern. Figure 1.302 shows the sets of fixed points for each of the four mappings, \( w_i \) where \( i = 1, 2, 3, 4 \). These sets correspond to the coloured sections seen in figure 1.302 i.e. red (\( w_1 \)), blue (\( w_2 \)), green (\( w_3 \)) and purple (\( w_4 \)). The Fundamental Theorem of Iterated Function Systems tells us that the union of these fixed points form the attractor/limit set of the IFS.

For IFSs in \( \mathbb{R}^2 \) there are two main algorithms used to visualise the attractor.

**Definition 1.304 - Deterministic Algorithm** (Barnsley, 1988)

Let \( \{X : w_1, w_2, \ldots, w_N\} \) be an IFS. Choose a compact set \( A_0 \subset \mathbb{R}^2 \). Then compute

\[
A_n = W^n(A) \quad \text{according to}
\]

\[
A_{n+1} = \bigcup_{j=1}^N w_j(A_n) \quad \text{for} \quad n = 1, 2, 3, \ldots
\]

Thus construct a sequence \( \{A_n : n = 0, 1, 2, \ldots\} \subset \mathcal{H}(X) \). Then by Theorem 1.301 the sequence \( \{A_n\} \) converges to the attractor of the IFS in the Hausdorff metric.
An alternate method of constructing an approximation to the attractor of an IFS is called The Random Iteration Algorithm, which utilizes randomness to create the attractor of an IFS as detailed below.

**Definition 1.305 - The Random Iteration Algorithm** (Barnsley, 1988)

Let \( \{X : w_1, w_2, ..., w_N\} \) be an IFS where probability \( p_i > 0 \) has been assigned to \( w_i \) for \( i = 1, 2, ..., N \), where \( \sum_{i=1}^{N} p_i = 1 \). Choose \( x_0 \in X \) and then choose recursively, independently,

\[
x_n \in \{w_1(x_{n-1}), w_2(x_{n-1}), ..., w_N(x_{n-1})\} \quad \text{for} \quad n = 1, 2, 3, ...
\]

where the probability of the event \( x_n = w_i(x_{n-1}) \) is \( p_i \). Thus construct a sequence \( \{x_n : n = 0, 1, 2, 3, ...\} \subset X \).

Barnsley proves that the sequence \( \{x_n\}_{n=1}^{\infty} \) fills in the attractor of the IFS densely.

When applying the contraction maps of the IFS it is possible to predict which maps produce a greater proportion of the attractor. In the case of figure 1.302 the stem of the fern requires the least amount of points to determine its eventual shape as opposed to the leaves. By applying probabilities to the contraction maps a general form of the attractor will be found. Adjusting the probabilities can alter the overall effectiveness of the algorithm. The mappings used to construct figure 1.302 have the following probabilities \( w_1 = 1\% \), \( w_2 = 85\% \), \( w_3 = 7\% \) and \( w_4 = 7\% \). If the probabilities are shared equally then the process will still produce the limit set, all be it slowly.

In comparing the two algorithms Barnsley (1988) states:

“The Random Iteration Algorithm can be applied to the computation of two-dimensional computer graphics. It has benefits compared to deterministic iteration of low memory requirements, high accuracy - as the iterated point can be kept at much higher precision than the resolution of the computed
image - and it allows the efficient computation of zooms into small parts of an image. However, as in the case of Deterministic Algorithms, the image produced depends on the computational details of image resolution, the precision to which the points \( \{x_1, x_2, \ldots, x_l\} \) are computed, the contractivity of the transformations, the way in which The Random Iteration Algorithm is implemented etc. Different implementations can produce different results.”

Given an IFS we know that an attractor will be produced, and once produced it can be analysed. The ‘inverse’ problem is, given a limit set/attractor/fractal set can an IFS be found which will produce it? If a set does not display self-similarity or is not self-affine then it may not be possible to find an IFS to produce an appropriate attractor (Falconer, 2003). In general, if an exact attractor cannot be created then an IFS which gives an approximate attractor can be found and the approximate attractor can be used for the purpose of fractal analysis. Further analysis of this inverse problem can be found in Barnsley (1988), Falconer (2003) and Frame et al (2005).

§4 Dimension Calculation

When using an iterative mapping system, such as an IFS, a limit set or attractor is formed which may exhibit fractal properties. Falconer (2003) suggested that calculating the fractal dimension and topological dimension of a set may help us classify it as either fractal or non-fractal.

The term fractal dimension is often equated to the Hausdorff-Besicovitch dimension. The Hausdorff-Besicovitch form of dimension calculation uses a set’s topological properties and can be applied to all sets contained within an arbitrary metric space. In order to calculate the Hausdorff-Besicovitch dimension the Hausdorff measure must first be calculated.

Assume the set \( A \) is embedded in the Euclidean space \( \mathbb{R}^n \) where the standard Euclidean metric holds; that is, the distance between any two points \( x \) and \( y \) is

\[
\sum_{i=0}^{d} (x_i - y_i)^2.
\]

Given this metric a measure of diameter can be applied to the set \( A \). Peitgen et al (1993) defined this measure as:
\[ \text{diam}(A) = \sup \{d(x, y) \mid x, y \in A\} \]

The set \( A \) can be covered by open balls/disks. A family of open subsets \( A_i \) where \( i = 1, 2, 3, \ldots \) will be an open (countable) covering of \( A \) only when \( A \subseteq \bigcup_{i=1}^{\infty} A_i \).

Let \( d \) and \( \varepsilon \) be positive real numbers.

Define

\[ h^d(\varepsilon)(A) = \inf \left\{ \sum_{i=0}^{\infty} \text{diam}(A_i)^d \mid \{A_i, \ i = 1, 2, \ldots\} \text{ is an open cover of } A \text{ with } \text{diam}(A_i) < \varepsilon \right\} \]

as \( \varepsilon \to 0 \) the following limit is formed

\[ h^d(A) = \lim_{\varepsilon \to 0} h^d(\varepsilon)(A). \]

**Definition 1.401 - \( d \)-dimensional Hausdorff Measure** (Peitgen et al, 1993)

Given \( A \), a subset of \( \mathbb{R}^n \), the limit \( h^d(A) = \lim_{\varepsilon \to 0} h^d(\varepsilon)(A) \) is called the \( d \)-dimensional Hausdorff measure of \( A \).

Hausdorff (1918) proved that for a set \( A \) there exists a unique real positive number \( D_H(A) \) which satisfies the following conditions:

\[ h^d(A) = \begin{cases} \infty & \text{for } d < D_H(A) \\ 0 & \text{for } d > D_H(A) \end{cases} \]

If \( d = D_H(A) \) then \( h^d(A) \) may be zero, infinite or some positive real number.

Intuitively, the \( d \)-dimensional Hausdorff measure indicates the size of the diameter a cover needs to provide a measure of a set. When \( d < D_H(A) \) the (diameter)^\( d \) is an ineffective cover for the set. In effect the open balls/disks are too small to cover
the set. When \( d > D_H(A) \) the (diameter)\(^d\) is still an ineffective cover for the set as the open balls/disks are too big to capture the detail of the given set, and hence give an accurate measure. When \( d = D_H(A) \) an appropriate covering can be made.

**Definition 1.402 - Hausdorff Dimension** \( (D_H(A)) \) (Peitgen et al, 1993)

This number \( D_H(A) \) is defined as the Hausdorff dimension

\[
D_H(A) = \inf \{d \mid h^d(A) = 0 \} = \sup \{d \mid h^d(A) = \infty \}
\]

The above definition is often referred to as the Hausdorff-Besicovitch dimension since Abram Samoilovitch Besicovitch computed the Hausdorff dimension for some highly irregular sets (Falconer, 2003).

**Example:**

The Hausdorff dimension of The Cantor Middle Third Set (C) is calculated, via first principles, as follows:

\[
C \subseteq \mathbb{R}. \ C \text{ can be covered very efficiently by } 2^n \text{ closed disks of diameter } \frac{1}{3^n}.
\]

Let \( \varepsilon = \frac{1}{3^n} \text{ for } n = 1, 2, 3, \ldots \) be a \( \varepsilon \)-cover for \( C \).

\[
h^d_{\varepsilon}(C) = \inf \left\{ \sum_{i=0}^{\infty} \text{diam}(A_i)^d \mid \{A_i, \ i = 1,2,\ldots \} \text{ is an open cover of } A \text{ with diam}(A_i) < \varepsilon \right\}
\]

\[
h^d_{\varepsilon}(C) = \frac{2^n}{3^{\frac{n^2}{3^n}}} \text{ for } n = 1, 2, 3, \ldots
\]

\[
h^d(C) = \lim_{\varepsilon \to 0} h^d_{\varepsilon}(C)
\]

\[
= \lim_{n \to \infty} \frac{2^n}{3^{\frac{n^2}{3^n}}}
\]
\( h^d (C) = \infty \) if \( d < \frac{\ln 2}{\ln 3} \)

\( h^d (C) = 1 \) if \( d = \frac{\ln 2}{\ln 3} \)

\( h^d (C) = 0 \) if \( d > \frac{\ln 2}{\ln 3} \)

Hence, \( D_H (C) = \frac{\ln 2}{\ln 3} \).

Other forms of fractal dimension calculation exist (Barnsley, 1988; Falconer, 2003 and Peitgen et al, 1993), such as box-counting dimension and self-similarity dimension. Not all forms of fractal dimension can be applied to all sets and when two differing forms of fractal dimension are used they may not produce the same results. For sets which display strict self-similarity, the self-similar dimension proves a far easier form of fractal dimension calculation e.g. when applied to sets such as The Cantor Middle Third Set and The Koch Curve.

**Definition 1.405 - Self-Similarity Dimension (\( d_s \))**

For a strictly self-similar set the self-similarity dimension, \( d_s \), is given by

\[
d_s = \frac{\ln(N)}{\ln(s)}
\]

where \( N \) is the number of copies of the set, scaled by a factor of \( \frac{1}{s} \) that are needed to cover the set.

If a set does not display strict self-similarity one of the alternate forms of fractal dimension calculation must be applied.

Fractal dimension provides a means by which to determine if a set is fractal by comparing its fractal dimension to its topological dimension (criterion point four). For completeness we will define a set’s topological dimension as follows:
Definition 1.406 - Topological Dimension \( (d_f) \) (Sutherland, 2002)

A set is \( n \)-dimensional if \( n \) independent variables are needed to describe a neighbourhood of any point.

Comparing a set’s fractal and topological dimension is not sufficient to classify a set as fractal or non-fractal. An example of this is the space-filling curve developed by Peano. The first three pre-fractals for this curve can be seen in figure 1.401, the initial set or \( 0^\text{th} \) pre-fractal is a line segment. Due to its construction the resulting set is both a closed two dimensional object and an infinite curve. This curve has both fractal and topological dimension two and hence fails criterion point four, yet is deemed fractal.

![Figure 1.401: Peano’s Curve (Weisstein, 2005)](image)

Frame et al (2005) use box-counting dimension as a means of calculating the fractal dimension of limit sets formed under iterative mapping systems, such as IFSs. Whereas the Hausdorff-Besicovitch dimension benefits from knowledge of the structure and construction method of a set the box-counting dimension requires only the resulting set. In many practical applications it is not possible to know the fine structure of a given set, this is true in relation to the fractal limit set created via an iterative process. In order to calculate the box-counting dimension we follow the following process.

Definition 1.406 - Box-Counting Dimension \( (d_b) \) (Roberts, 2003)

Given a set from within \( \hat{\mathbb{R}}, \hat{\mathbb{R}}^2 \) or \( \hat{\mathbb{R}}^3 \), place a grid with boxes (intervals, squares or cubes) of size \( x \) over the set. Count the number \( (N(h)) \) of boxes that contain elements of the set. Repeat this process with boxes scaled by \( \frac{1}{h} \). Plot a \( \ln (N(h)) \) verse \( \ln \left( \frac{1}{h} \right) \) graph.
and find the gradient of the line of best fit, call it $m$. The box-counting dimension is defined to be the slope of the line of best fit:

$$d_b = m$$

**Example:** (Elert, 2005)

Figure 1.402 shows the calculation of the box-counting dimension for The Koch Curve. This is an accurate measure for the set after limited applications (only 6% deviation after 12 applications).

Using self-similarity dimension we obtain

$$d_s = \frac{\ln 4^n}{\ln 3^n} = \frac{\ln 4}{\ln 3} \approx 1.26$$

Using box-counting dimension Elert (2005) obtains:
With advancement in technology the box-counting dimension can be found via automated computer programming.

For example Bourke (2003) provides details about a program called Fractal Dimension Calculator (FDC).

“FDC estimates the fractal dimension of an object represented as a black and white image where the object to be analysed is assumed to be made up of the black pixels. This is accomplished by an algorithm called ‘box-counting’.”

Like many aspects of computer programming the technology is rapidly improving.
§5 Circle Inversion

Frame, Mandelbrot and Neger (2005) created fractal limit sets using iterative mapping systems within the extended real plane \( \mathbb{R}^2 \). These iterative systems comprised of collections of circle inversion mappings.

The following geometric definition of circle inversion dates back to the time of Apollonius of Perga (b.262BC) and is applied to given circles in the metric space \( \mathbb{R}^2 \).

**Definition 1.501 - Inversion** (Frame et al, 2005)

Given a point \( P \) outside the circle \( C \), draw the segment \( PQ \) tangent to \( C \). From \( Q \) drop the perpendicular to \( OP \) intersecting at \( P' \). The point \( P' \) is the inverse of \( P \). Conversely, given a point \( P' \) inside the circle \( C \), draw the perpendicular to \( OP' \) intersecting at \( Q \). The tangent to \( C \) at \( Q \) intersects the line extending \( OP' \) at \( P \). The point \( P \) is the inverse of \( P' \).

![Figure 1.501 - Inversion (Frame et al, 2005)](image)

Algebraically, the definition for inversion of a point is:

**Definition 1.502 - Inversion of a Point** (Frame et al, 2005)

\( P' \) is the inverse of \( P \) if

(i) \( P \) and \( P' \) lie on the same ray from \( O \)

and

(ii) \( OP \times OP' = OQ^2 \)
From the above definition it can be shown that inversion across the circle $C$, denoted $i_C$, where circle $C$ has centre $(a, b)$ and radius $r$ can be written as:

$$(x', y') = i_C(x, y) = (a, b) + \frac{r^2}{(x - a)^2 + (y - b)^2}(x - a, y - b)$$

where $P = (x, y)$ and $P' = (x', y')$

![Diagram of inversion](image)

Figure 1.502: Inversion of a point (Frame et al, 2005)

In this formulation $i_C$ is of the form $i_C : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Inversion can also be thought of as a map $i_C : \hat{C} \rightarrow \hat{C}$, where $\hat{C} = \mathbb{C} \cup \infty$ represents the extended complex plane.

**Definition 1.503 - Inversion (Complex Plane)** (Needham, 2000)

Inversion ($i_K$) of a point $z \in \hat{C}$ across an arbitrary circle $K$ of centre $q$ ($q \in \hat{C}$) and radius $r$ is defined as:

$$i_K(z) = \frac{r^2}{z - q} + q$$

where

$$i_K(q) = \infty$$

and

$$i_K(\infty) = q$$

Inversion of a point $z \in \hat{C}$ in the unit circle is defined as:
where
\[ i_u(z) = \frac{1}{z} \]

\[ i_u(0) = \infty \]

and
\[ i_u(\infty) = 0 \]

The following are some of the properties of circle inversion which can be applied to points within the metric spaces \( \mathbb{R}^2 \) and \( \hat{\mathbb{C}} \). These properties are expressed in terms of points/circles/lines contained within \( \mathbb{R}^2 \). Properties and associated formulae analogous to those shown here can be found for points/circles/lines contained within \( \hat{\mathbb{C}} \).

**Property 1.501 - Inversion of a Circle** (Frame et al, 2005)

(a) The circle \( S \) with centre \((c, d)\) and radius \(s\) and which does not pass through \((a, b)\) will be inverted across circle \( C \) with centre \((a, b)\) and radius \(r\) to a circle \( S' \) with centre \((u, v)\) and radius \(R\). The newly inverted circle can be found via the following formulas:

\[
(u, v) = (a, b) + \frac{r^2}{(c - a)^2 + (d - b)^2 - s^2} (c - a, d - b) \quad \text{………… (1)}
\]

\[
R = \frac{r^2 s}{(c - a)^2 + (d - b)^2 - s^2} \quad \text{………… (2)}
\]

**Note**: The centre \((u, v)\) of the newly inverted circle is, in general, not the inverse of the centre of \( S \). For the centre \((u, v)\) of the newly inverted circle to coincide with the centre of circle \( S \) the circles \( C \) and \( S \) must be concentric.

(b) The circle \( S \) with centre \((c, d)\) and radius \(s\) and which does pass through \((a, b)\) will be inverted to a line.
**Property 1.502 - Circumference Points** (Frame et al, 2005)

Inversion across a circle $C$ with centre $(a, b)$ leaves unchanged every point on the circle $C$.

**Property 1.503 - Interior/Exterior Exchange** (Frame et al, 2005)

Inversion across a circle $C$ with centre $(a, b)$ interchanges the inside and outside of the circle $C$.

**Property 1.504 - Contraction** (Frame et al, 2005)

Inversion across a circle $C$ with centre $(a, b)$ is a contraction on sets outside the circle $C$.

**Note:** Points on the circumference of $C$ are left invariant under inversion across $C$ and as such does not exhibit a contractivity factor strictly less than one (Definition 1.202). Hence Property 1.504 does not include points on the circumference of $C$. Inversion across a circle $C$ with centre $(a, b)$ is only a contraction on sets which are exterior to $C$.

**Property 1.505 - Orthogonal Circle 1** (Frame et al, 2005)

Inversion across a circle $C$ with centre $(a, b)$ preserves circle $S$ orthogonal to the inverting circle $C$.

**Property 1.506 - Involution** (Frame et al, 2005)

Inversion across a circle $C$ with centre $(a, b)$ is an involution. That is, if $(x', y')$ is the inverse of $(x, y)$ across $C$, then $(x, y)$ is the inverse of $(x', y')$ across $C$.

**Property 1.507 - Orthogonal Circle 2** (Pedoe, 1988)

If $S$ is any circle which passes through a pair of points $P$ and $P'$, where $P'$ is the image of $P$ under inversion across circle $C$, then $S$ is orthogonal to $C$. 
Property 1.508 - Conformal Mapping (Pedoe, 1988)

Inversion is a mapping that does not alter the angle of intersection of two circles.

Property 1.509 (Perfect, 1963)

A straight line $L$ that does not pass through $(a, b)$ (the centre of the circle of inversion) inverts with respect to $C$ into a circle through $(a, b)$.

§6 Möbius Transformations

The idea of studying limit sets of iterated maps has been extended by Mumford, Series and Wright (2002). In their book, *Indra’s Pearls: The Vision of Felix Klein*, Mumford et al (2002) introduced the idea of repeated iteration of Möbius transformations in the extended complex plane $\hat{\mathbb{C}}$.

Definition 1.601 - Möbius Transformations (Blair, 2000)

A Möbius transformation (also called a linear fractional transformation) is a transformation of the form:

$$M(z) : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$$

$$M(z) = \frac{az + b}{cz + d} \quad \text{where } a,b,c,d \in \mathbb{C}, \ ad - bc \neq 0$$

and

$$M\left(\frac{d}{c}\right) = \infty$$

$$M(\infty) = \frac{a}{c}$$

The theory of Möbius transformations is well established. Many texts exist outlining this theory, see for example *Visual Complex Analysis* by Needham (1997).
Some standard results that will be used include the following properties:

- Every Möbius transformation is a conformal mapping, but the reciprocal is untrue.
- A Möbius transformation is a bijection on $\hat{\mathbb{C}}$.
- A Möbius transformation can be composed from the elementary transformations of dilations, translations, rotations and inversions.

In order to visualize a Möbius transformation stereographic projection onto the Riemann Sphere is often used, an example of such a projection is seen below.

![Figure 1.601 - Westendorp (2006)](image)

**Example:** (Wikipedia and Weisstein, 2007)

A classic example of a Möbius transformation is The Cayley Transform. The Cayley Transform is given below:

$$M(z) = \frac{z - i}{z + i}$$

$M(z)$ takes the upper half-plane $\text{Im}(z) > 0$ conformally onto the unit disk $D: |z| < 1$ as shown in figure 1.602

![Figure 1.602 - Wikipedia (2008)](image)
When dealing with a Möbius transformation, \( M(z) \), it is convenient to express it in matrix form as it allows us to relate map composition to the operation of matrix multiplication.

\[
M(z) = \frac{az + b}{cz + d} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

Multiplying \( M(z) \) by any complex number \( k \) does not alter the map \( M(z) \). That is

\[
M(z) = \frac{az + b}{cz + d} = \frac{kaz + kb}{kcz + kd}.
\]

A value for \( k \) can be chosen such that the associated matrix has a determinant equal to one, \( \det(M) = 1 \). This can be achieved by normalizing the matrix associated with a given Möbius transformation. For the purpose of this research we will assume all matrices are normalised unless otherwise stated.

Iteration of a single Möbius transformation does not result in any fractal limit sets. However, the ideas and results of the iterations of these mappings are important for the analysis of more complicated iterative systems.

Note: \( tr(M) \) is used to denote the trace of \( M \) i.e. the trace of the matrix associated with the Möbius transformation \( M(z) \).

Iteration of a single Möbius transformation produces the following classification:

1. **Parabolic**

**Definition 1.602** (Wikipedia, 2006)

A Möbius transformation is called parabolic if:

\[
(tr(M))^2 = (a + d)^2 = 4
\]

i.e. \( tr(M) = \pm 2 \)
A Möbius transformation is parabolic if a unique fixed point exists which is both a source and a sink. Under a parabolic mapping points move in a circular motion, i.e. have circular orbits as shown in figure 1.603.

![Figure 1.603 - Behnert (2006)](image)

2. **Elliptic**

**Definition 1.603** (Wikipedia, 2006)

A Möbius transformation is called elliptic if:

\[ tr(M) \in \mathbb{R} \]

and

\[ -2 < tr(M) < 2 \]

An elliptic map will have two fixed points. These fixed points will be neutral (i.e. no sink and/or no source) and points move in circles separating the fixed points, as shown in figure 1.604.

![Figure 1.604 - Behnert (2006)](image)
3. **Loxodromic**

**Definition 1.604** (Wikipedia, 2006)

A Möbius transformation is called loxodromic if:

\[ tr(M) \in \mathbb{C} \]
and
\[ tr(M) \notin [-2, 2] \]

A loxodromic map will have two fixed points, one sink and one source. Points will spiral from the source into the sink.

4. **Hyperbolic (special case of Loxodromic)**

**Definition 1.605** (Wikipedia, 2006)

A Möbius transformation is called hyperbolic if:

\[ tr(M) \in \mathbb{R} \]
and
\[ tr(M) \notin [-2, 2] \]

A hyperbolic map will have two fixed points. Hyperbolic maps are a special case of loxodromics in which points move not in spirals but in a circular motion, as shown in figure 1.606.
In the theory of Möbius transformations conjugate maps are used. In this context conjugation is an algebraic construct of the form $\hat{M} = SMS^{-1}$ where $M$ and $S$ are Möbius transformations expressed in normalised matrix form. The transformation $M'(z)$ exhibits all the same features as the transformation $M(z)$ just moved to a new position by the transformation $S(z)$. A given Möbius transformation $M(z)$ and the conjugate transformation $\hat{M}(z)$ behave the same geometrically since conjugation does not change the trace of the associated matrix. In effect, the conjugation of Möbius transformations allows us to view transformations in terms of a canonical form.

Conjugation allows us to study each type of Möbius transformation by considering only the canonical form as shown in the following table.

<table>
<thead>
<tr>
<th>Type of Transformation</th>
<th>Fixed Point (s)</th>
<th>Canonical Form</th>
<th>Type of Mapping</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parabolic</td>
<td>$\infty$</td>
<td>$M(z) = z + a$, $k = 1$</td>
<td>Translation</td>
</tr>
<tr>
<td>Elliptic</td>
<td>0, $\infty$</td>
<td>$M(z) = kz$, $</td>
<td>k</td>
</tr>
<tr>
<td>Hyperbolic</td>
<td>$y_1$, $y_2$</td>
<td>$M(z) = kz$, $k &gt; 1$, $k \in \mathbb{R}$</td>
<td>Scaling Maps</td>
</tr>
<tr>
<td>Loxodromic</td>
<td>$y_1 \neq y_2$</td>
<td>$M(z) = kz$, $</td>
<td>k</td>
</tr>
</tbody>
</table>

Figure 1.606 - Behnert (2006)
When dealing with hyperbolic and loxodromic maps the fixed points, \( y_1 \) and \( y_2 \), will be either attracting (sink) or repelling (source). In order to determine their nature the following formula is used:

\[
k = \frac{a - y_1}{c - y_2}, \quad y_1 \neq y_2
\]

\[
\frac{2}{4 - (tr(M))^2}
\]

If \(|k| > 1\) then
\[
\begin{cases} 
  y_1 \text{ is attracting (sink)} \\
  y_2 \text{ is repelling (source)}
\end{cases}
\]

If \(|k| < 1\) then
\[
\begin{cases} 
  y_2 \text{ is attracting (sink)} \\
  y_1 \text{ is repelling (source)}
\end{cases}
\]

Of particular interest are those Möbius transformations which Mumford et al (2002) describes as ‘pairing maps’. A given transformation pairs two circles if the transformation takes the interior of one circle and maps it to the exterior of the other and visa versa.

In figure 1.607 \( A \) maps the exterior of circle \( C_1 \) to the interior of circle \( C_2 \). The map \( a \) is the inverse of map \( A \), and as such takes the exterior of circle \( C_2 \) to the interior of circle \( C_1 \).

\[
A : \text{exterior } C_1 \mapsto \text{interior } C_2
\]

\[
a : \text{exterior } C_2 \mapsto \text{interior } C_1 \quad \text{where } A = a^{-1}
\]
Circle pairing maps can be found using a systematic approach. We begin by selecting two circles $C_1$ and $C_2$ with centres $P$ and $Q$ and radii $r$ and $s$ respectively.

![Diagram of two circles with centres P and Q and radii r and s](image)

Figure 1.608

Pairing maps are not unique. Mumford et al (2002) describe how to create such mappings. Circle pairing mappings can be expressed in terms of both radii and centres of our given circles (figure 1.608) and also contain variables $u$ and $v$, as follows:

$$M(z) = s \frac{u(z - P) + ru}{u(z - P) + rv} + Q.$$  

The variable $u$ and $v$ are chosen such that they satisfy the equation $\mid u \mid^2 - \mid v \mid^2 = 1$. If however a specific point is to be mapped to another the above transformation can be altered by careful choice of either $u$ or $v$.

Mumford et al (2002) state:

“By judiciously choosing the values $u$ and $v$, you can move 0 to any point on the unit disk: first stretch by choosing $u$ and $v$ real so that 0 gets moved to a chosen point $\frac{v}{u}$ inside the unit disk and on the real axis, then rotate about 0 until your point reaches the required position in the unit disk”.

Mumford et al (2002) further state that given the circle pairing mappings shown above will:
“carry the inside of any circle $C$ to the outside of any other circle $C'$, with the additional requirement that some given point inside $C$ maps to the point $\infty$ outside $C'$.

Letting $u = 1$ and $v = 0$ a circle pairing map is found which requires only knowledge of the given circles centres and radii, as shown below:

$$M(z) = \frac{rs}{z - P} + Q.$$ 

Note that the mapping $i_k(z) = \frac{r^2}{z - q} + q$, which represents inversion in $\hat{C}$, is not a circle pairing mapping nor is it a Möbius transformation (Definition 1.601) as this mapping is anti-conformal (Needham, 2000).
Chapter 2 Circle Inversion Iteration

In Chapter 1 §3 we examined the work of Barnsley (1988) and the construction of fractal limit sets using IFSs. In this chapter we will use Barnsley’s (1988) process of repeated iteration and apply it to collections of circle inversion mappings, denoted \( i_{C_i} \), (Definition 1.502). Using collections of such mappings we will develop iterative systems which we will term Iterated Circle Inversion Systems (ICISs).

**Iterated Circle Inversion System (ICIS)/ \( n^{th} \) order ICIS**

An Iterated Circle Inversion System (ICIS) is a finite collection \( \{ i_{C_i} \}_{k=1}^n \) of circle inversion mappings \( i_{C_i} : \mathbb{R}^2 \to \mathbb{R}^2 \), where the \( C_i \)'s are distinct.

An ICIS is not an IFS. Definition 1.301 – Iterated Function Systems states that IFSs require a complete metric space \( (X, d) \) and a finite set of contraction mappings \( w = \{ w_i \}_{i=1}^N \) where \( w_i : X \to X \). ICISs can be applied to sets contained within the complete metric space \( \hat{\mathbb{R}}^2 \), the extended real plane, yet the mappings \( i_{C_i} \) are not contraction mappings for all sets within \( \hat{\mathbb{R}}^2 \). Property 1.504 – Contraction states that the mappings \( i_{C_i} \) are only contraction mappings for sets exterior to the circle of inversion. Hence the mappings \( i_{C_i} \) are contraction mappings only on restricted domains. A (contractive) circle inversion mapping takes the exterior of the circle of inversion and maps it into the interior of the circle of inversion, hence no ICIS satisfies Definition 1.301.

Even though ICISs do not constitute IFSs mathematicians such as Frame and Cogelevina (2000), Mandelbrot and Frame (2002) and Frame, Mandelbrot and Neger (2005) apply the mechanics of IFSs to the study of circle inversion iteration. In order to apply such principles we will need to place restrictions on the initial circle placements and restrict the domain of our inversion mappings \( i_{C_i} \) to ensure the mappings \( i_{C_i} \) are never expansion mappings, that is, no mappings \( i_{C_i} \) has a scaling factor \( s \) where \( s > 1 \). Within this chapter we will follow the work of Frame et al (2005) and restrict ourselves...
to non-overlapping circle placements, these being circle placements which have disjoint interiors and at most one point of contact between the boundaries of the respective circles of inversion. An ICIS with such restrictions applied to it will be termed a restricted ICIS. Iteration of a restricted ICIS will proceed as follows:

Consider a set of two or more non-overlapping circles \( \{C_k\}_{k=1}^n \). Let \( R_0 \) represent the region which lies exterior to circles \( \{C_k\}_{k=1}^n \).

1. Begin with the initial set \( R_0 \) in \( \mathbb{R}^2 \).
2. Invert \( R_0 \) into all the \( C_k \)'s.
3. Label the resulting inversion sets relative to the circle of inversion i.e. after inverting into circle \( C_n \) the resulting set is termed \( R_0_n \).
4. Invert the sets \( R_0_k : k = 1, \ldots, n \) into the \( C_k \)'s with the restriction that \( R_0_n \) may not be inverted into circle \( C_n \) \( \forall n \). This produces a further inversion set \( R_{0_{kj}} : k, j = 1, \ldots, n \) with \( k \neq j \).
5. Repeat this process ad infinitum.

We will study the limit sets, denoted \( \mathcal{L} \), of these sequences and show how fractal limit sets can be formed under iteration of restricted ICISs. Note that after one iteration any set wholly contained within \( R_0 \) will have its inversion image set wholly contained within \( \{C_k\}_{k=1}^n \). Furthermore, points on the boundary of \( \{C_k\}_{k=1}^n \) remain invariant under iteration. Hence we need only examine iterations of circles \( \{C_k\}_{k=1}^n \).

We will show that a restricted ICIS will, under iteration, converges to a limit set and that this limit set is unique to the initial circle placement. The existence of unique limit sets for a given restricted ICIS will allow us to classify the structure of limit sets produced under restricted ICISs.

§1 Restricted Iterated Circle Inversion Systems (Restricted ICISs)

The property of convergence underpins the work of Barnsley (1988) and Hart (1996) in relation to IFSs. An IFS utilizes contraction mappings and given any initial set repeated application of an IFS will produce a sequence of subsets that converge toward
a unique fixed subset, the attractor of the IFS. Barnsley (1988) and Hart (1996) used the Hutchinson operator and the Hutchinson Lemma to prove a unique attractor via Theorem 1.301- The Fundamental Theorem of IFSs. This approach is not applicable to restricted ICISs as the Hutchinson Lemma requires a finite set of strictly contractive mappings applied to a complete metric space. Inversion mappings $i_{C_i}$ are not strictly contractive for all sets in $\mathbb{R}^2$ since a circle placement which has one or more pair of tangential circles has associated scaling factors, $s$, of mappings $i_{C_i}$ such that $0 \leq s \leq 1$.

Under iteration a restricted ICIS converges to a limit set. As previously stated we begin the inversion iteration process by considering the set of circles $\{C_k\}_{k=1}^n$. Property 1.501 – Inversion of a Circle tells us that under inversion a circle inverts to another circle. If the initial circle placement is totally disjoint then the resulting inversion image circles are wholly contained within $\{C_k\}_{k=1}^n$. If there exists one or more pair of tangential circles the image circles are contained either wholly within or contained within with point(s) of intersection on the boundary of $\{C_k\}_{k=1}^n$. That is if a placement has a pair of tangent circles, $C_i$ and $C_j$, the point of tangency is contained within the boundary of both $C_i$ and $C_j$ and remains invariant under iteration due to Property 1.502 – Circumference Points. Throughout the iteration process all points, excluding any points of tangency, are inverted using strictly contractive inversion mappings. This results in a nested set of inversion image circles being formed. The length of the radii of these image circles decrease throughout the iteration process converging to zero, forming a collection of limit points. The union of these limit points is termed the limit set of the restricted $n^{th}$ order ICIS.

Example:

A restricted 2$^{nd}$ order ICIS applied to two non-overlapping circles $A$ and $B$ consists of two cases. Firstly, if $A$ and $B$ are totally disjoint then the limit set consists of two points. These two points are the fixed set of the restricted 2$^{nd}$ order ICIS, a proof of this can be found in Lalley (1994). If however $A$ and $B$ have a point of intersection then the limit set consists of the point of intersection exclusively (Frame et al (2005) and
Let $AB$ represent the inversion of $B$ in $A$ and $BA$ represent the inversion of $A$ in $B$. Figure 2.101 shows that if circles $A$ and $B$ are totally disjoint then circle $A$ contains all subsequent inversions in $A$ and likewise for circle $B$. Since the circles decrease in size throughout the inversion process, the eventual limit set will consist of only two points.

Figure 2.101: Two non-overlapping circles (Frame et al, 2005)

If the circles $A$ and $B$ are tangent then the limit set will be a single point, the point of tangency. Denote the point of tangency as $p_0$. Throughout the iteration process $p_0$ remains invariant and as such will be contained within the boundary of all inversion image circles. Throughout the inversion iteration process the diameter of the inversion image circles converge to zero. Consequently, the limit set consists of an isolated point, namely $p_0$.

The mappings $i_{c_1}$, on restricted domains, are unique to the respective circles of inversion (Definition 1.502 – Inversion of a Point). All limit points contained within the limit set, $\mathcal{L}$, are a result of the inversion image circles forming sets of nested circles, the diameter of which converge to zero under iteration. Since at all levels of iterations the inversion image circles are unique to the initial circle placement all limit points are unique to the initial circle placement and fixed under circle inversion iteration. Hence the restricted $n^{th}$ order ICIS limit set, $\mathcal{L}$, is unique to the initial circle placement and fixed under circle inversion iteration.

Restricted 2$^{nd}$ order ICISs only produce non-fractal limit sets. Fractal limit sets require a minimum of three disjoint non-overlapping circles. That is, fractal limit sets can be formed under a restricted $n^{th}$ order ICIS for $n \geq 3$. 
Given three non-overlapping circles in the plane there exists a unique circle/line that cuts the three given circles at right angles. This circle/line is said to be orthogonal to the three original circles. This unique orthogonal circle/line can be computed via the solution to Monge’s Problem (see appendix). For a restricted 3\textsuperscript{rd} order ICIS the resulting limit set and the three circle placements unique orthogonal circle/line are closely linked.

Mandelbrot (1983) observed “if the three non-overlapping circles are pairwise touching (tangent) the limit set will be the entire orthogonal circle and if not, then the limit set will be a fractal dust contained within the orthogonal circle”. That is the limit set of a restricted 3\textsuperscript{rd} order ICIS is either fractal or non-fractal. If the circle placement has at least two circles which are totally disjoint the resulting limit set will be a fractal dust, often termed a Cantor dust, contained within the unique orthogonal circle. If the circle placement consists of three tangential circles the limit set is the unique orthogonal circle.

It can be seen, in figures 2.102 – 2.106, that throughout the inversion iteration process, applied to a set of three tangential circles, the resulting inversion image circles form nested sets which converge to a non-fractal limit set. Start with a set of three pairwise tangent circles.

![Figure 2.102: Three pairwise tangent circles](image)

After one iteration we have six newly formed inversion image circles which form a tangent chain contained within the initial circles of inversion.

![Figure 2.103](image)
After two iterations we have a further twelve newly formed inversion image circles. It is important to note that at each level of iteration the newly formed inversion image circles form tangent chains. At no level of iteration does this tangent chain become disjoint; this is due to the mapping of points of tangency at all levels.

Throughout the inversion iteration process the newly inverted circles form tangent chains of nested sets of circles, the diameters of which are converging to zero. The tangent chain converges to a connected curve.

As shown, in figure 2.106, this connected curve is the circle orthogonal to the initial circle placement. The curve must be the unique orthogonal circle as any other curve would not remain invariant under circle inversion iteration.
Given a set of three non-overlapping circles if there exists at least one pair of circles which are not pairwise tangential then the resulting restricted 3rd order ICIS limit set will be a fractal dust contained on the unique orthogonal circle. Due to a disconnection in the initial circle placement disconnection between newly formed inversion image circles appear at all levels of iteration. This forms a fragmented set, a set of infinitely many distinct points lying on the unique orthogonal circle, as shown in figure 2.107.

![Figure 2.107](image)

When applying a restricted $n^{th}$ order ICIS to $n$ distinct non-overlapping circles multiple orthogonal circles can be formed. These orthogonal circles are formed by taking every combination of three circles in the placements, called 3-tuples. As we will see in §2 given $n$ distinct non-overlapping circles the number of orthogonal circles formed from all available 3-tuples as well as their relative placement with respect to the initial circle placement will enable us to predict the structure of the resulting restricted $n^{th}$ order ICIS limit set.

In order to analyse and classify the structure of restricted $n^{th}$ order ICIS limit sets we must first determine methods of construction. We begin with The Deterministic Algorithm (Definition 1.304) and The Random Iteration Algorithm (Definition 1.305). These algorithms can be used to visualise the attractor of an IFS. Since no ICIS satisfies Definition 1.301 – Iterated Function Systems we will need to find analogous methods. Frame and Cogevina (2000) defined variations of both The Deterministic Algorithm and The Random Iteration Algorithm utilising known fixed sets, the unique orthogonal circles/lines, as the initial set. The Deterministic Algorithm given by Frame and Cogevina (2000) can be applied to all restricted $n^{th}$ order ICISs where $n > 2$.

The following are the two main algorithms which enable us to visualise the limit set of a restricted $n^{th}$ order ICIS. These algorithms are called The Deterministic
Inversion Algorithm for a Restricted $n^{th}$ order ICIS and The Random Inversion Algorithm for a Restricted $n^{th}$ order ICIS. When applying The Deterministic Inversion Algorithm for a Restricted $n^{th}$ order ICIS the number of points formed by each iteration grow exponentially i.e. starting with $n$ distinct circles and a single point $p$ the number of points produces after $k$, $k \geq 1$, iterations will be $n(n-1)^{k-1}$.

**Definition 2.101 - The Deterministic Inversion Algorithm for a Restricted $n^{th}$ order ICIS** (Frame and Cogevina (2000))

1. Given non-overlapping inverting circles $\{C_k\}_{k=1}^n$ generate all $\binom{n}{C_j}$ orthogonal circles $S_1,...,S_k$. The $S_j$’s belong to the limit set.
2. Invert each $S_j$ in all the $C_i$’s not orthogonal to it.
3. Invert in all $C_i$’s, the circles, the $S_j$’s generated in the previous step, with the conditions that the same $C_i$ is not selected twice successively and a circle is not inverted in a circle it intersects orthogonally.
4. Repeat 3 until some preset termination condition has been reached.

**Definition 2.102 - The Random Inversion Algorithm for a Restricted $n^{th}$ order ICIS** (Frame and Cogevina (2000))

1. Given non-overlapping inverting circles $\{C_k\}_{k=1}^n$ define a starting point $(x_0,y_0)$ external to all the inverting circles.
2. Select a $C_j$ randomly and generate $(x_{n+1},y_{n+1})$ by applying to $(x_n,y_n)$ the point inversion formula (Definition 1.502) with $C = C_j$.
3. Repeat step 2, with the condition that the same $C_j$ is not selected twice successively. Plot the point $(x_{n+1},y_{n+1})$ for $n > 20$, allowing the orbit of $(x_0,y_0)$ to converge within a pixel of the limit set. Continue until some preset termination condition has been reached.
The algorithm presented in Definition 2.102 was used to create many of the images presented in this thesis. These images were created using the mathematical software Maple 8 (see appendix). The above algorithm (Definition 2.102) relies on The Random Iteration Theorem for a Restricted $n^{th}$ order ICIS.

**Theorem 2.101 - The Random Iteration Theorem for a Restricted $n^{th}$ order ICIS**

Consider the restricted $n^{th}$ order ICIS \( \{ \mathbb{R}^2, i_{c_k}, k = 1,2,\ldots,n \} \) where \( n \geq 3 \). Let \( x_0 \) be a limit point of one of the 2$^{nd}$ order restricted ICISs contained within the restricted $n^{th}$ order ICIS and let \( \{ x_i \} \) be a sequence of numbers chosen randomly from the set \( \{1,2,\ldots,n\} \) then:

i. All points in the sequence

\[
x_0, x_1 = i_{c_{x_1}}(x_0), x_2 = i_{c_{x_1}}(x_1),\ldots
\]

lie on the limit set, denoted by \( \mathcal{L} \), of the restricted $n^{th}$ order ICIS.

ii. The sequence

\[
x_0, x_1 = i_{c_{x_1}}(x_0), x_2 = i_{c_{x_1}}(x_1),\ldots
\]

(eventually) fills \( \mathcal{L} \) densely.

**Note:** Theorem 2.101 (ii) states that a sequence of points will fill \( \mathcal{L} \) densely. It is possible to have a sequence of points which alternate between a select number of initial circles, hence not producing the entire restricted $n^{th}$ order ICIS limit set. This possibility is highly improbably as it is assumed that all circles will be chosen at least once throughout the iteration process. Since this process continues ad infinitum this assumption is valid. Although not used within this thesis a probability statement can be added to Theorem 2.101 to justify the above mention assumption.
Proof:

i. $x_0$ is contained within the restricted $n^{th}$ order ICIS since $x_0$ is a limit point of a restricted $2^{nd}$ order ICIS limit set contained within the restricted $n^{th}$ order ICIS. Let the limit set of the restricted $n^{th}$ order ICIS be denoted by $\mathcal{L}$, we have $x_0 \in \mathcal{L}$. We must show that under any mapping, $i_{c_1}$, $x_0$ inverts to another limit point contained within $\mathcal{L}$.

Throughout the circle inversion iteration process nested sets of circles were formed from all possible combinations/iterations of mappings $i_{c_k}$ applied to $\{C_i \}_{k=1}^n$ as well as all resulting inversion image circles. All limit points are a result of diameters of the nested sets of inversion image circles decreasing (converging to zero) under iteration. That is every limit point is the limit of a specific sequence of mappings $i_{c_k}$. Given $n \geq 3$ non-overlapping circles there exist infinitely many such sequences formed from all possible combinations of mappings $i_{c_k}$. $x_0$ is the limit of one specific sequence of mappings. $i_{c_{n}}(x_0)$ is an infinite sequence of inversion mappings, since $x_0$ is itself the limit of an infinite sequence. Since the limit of all infinite sequence of mappings $i_{c_{n}}$ are limit points of the restricted $n^{th}$ order ICIS, $x_0$ inverts to another limit point contained in $\mathcal{L}$.

ii. Let $x_0$ be a limit point of one of the $2^{nd}$ order restricted ICISs contained within the restricted $n^{th}$ order ICIS. We must show that if the sequence

$$x_0, x_1 = i_{c_{n}}(x_0), x_2 = i_{c_{n}}(x_1), \ldots$$

is allowed to continue sufficiently far that it will produce a point arbitrarily close to $x_0$, i.e. for any $\varepsilon > 0$ there is a point $q$ in the sequence such that $d(x_0, q) < \varepsilon$. 
Since the restricted \( n^{th} \) order ICIS is defined on \( n \) distinct non-overlapping circles we can allocate an address to the point \( x_0 \). The address of \( x_0 \in \mathcal{Q} \) will be an infinite sequence

\[
\ldots c_1, c_2, \ldots, c_m, \ldots
\]

where \( c_1, c_2, \ldots, c_m, \ldots \) refers to successively nested circles (as shown in the 2\( \text{nd} \) order restricted ICIS example).

Consider the first \( m \) digits of the address of \( x_0 \). This refers to a sub-circle that contains \( x_0 \), call this sub-circle \( \lambda_m \). Since throughout the inversion process the diameter of the sub-circles decrease (converge to zero), the diameter of \( \lambda_m \) will satisfy \( \lambda_m < \varepsilon \) for a sufficiently large \( m \).

We can produce a sequence

\[
\alpha_1 \alpha_2 \ldots \alpha_k \ldots
\]

where \( \alpha_k \in \{1, 2, 3, \ldots, n\} \) are the random numbers chosen at each stage of the algorithm. If anywhere within this sequence the block

\[
c_m c_{m-1} \ldots c_2 c_1
\]

occurs than a point in \( \lambda_m \) will have been marked. This is because, regardless of where the point is after the \( k^{th} \) stage of the algorithm, the sequence of inversion maps referred to in the block will take the point to a point in \( \lambda_m \).

Given that the points in the sequence \( \alpha_1 \alpha_2 \ldots \alpha_k \ldots \) is generated randomly, the probability that a subsequence of length \( m \) that matches \( c_m c_{m-1} \ldots c_2 c_1 \) occurs is
\[ \frac{1}{n^m}, \] this is a finite number for any given circle placement. If the sequence is allowed to continue for long enough then a subsequence will occur and the algorithm will have produced a point \( q \) such that \( d(x_0, q) < \varepsilon \).

The above theorem states that The Random Inversion Algorithm for a Restricted \( n^{th} \) order ICIS will eventually draw the limit set (\( \mathcal{L} \)) of a given restricted \( n^{th} \) order ICIS. Through experimentation it can be seen that “any portion of the limit set near the points of tangency of the inverting circles will fill in very slowly under random iteration” (Frame and Cogevina, 2000). This problem can be overcome by assigning probabilities to the respective inversion mappings \( i_{C_i} \).

We will now discuss two alternate ways of generating (approximate) limit sets of restricted \( n^{th} \) order ICISs. Firstly we will follow the work of Frame et al (2005) in the construction of a restricted \( n^{th} \) order ICIS limit set. We will term this construction process, when applied to a restricted \( n^{th} \) order ICIS where \( n > 2 \), Frame’s Algorithm. Secondly we will follow the construction process developed by Mandelbrot (1982), termed Mandelbrot’s Algorithm.

**Frame’s Algorithm** (Frame et al, 2005)

Frame’s Algorithm is used when dealing with three or more non-overlapping circles in a given placement. This algorithm begins by utilising the known limit sets by determining the inversion limit set for each pair of non-overlapping circles (restricted 2\(^{nd}\) order ICIS limit set), which will constitute either a single point or two distinct points (Frame et al, 2005). This algorithm uses the fact that these limit sets are subsets of the overall restricted \( n^{th} \) order ICIS limit set since they would be formed by repeated iteration on only two circles. Since these points lie in the greater limit set, iteration of these points will produce further points which lie in the desired limit set. Frame’s Algorithm is in essence The Deterministic Algorithm with a wise choice of initial set.
In the following example we begin with three totally disjoint non-overlapping circles, the limit set of each restricted 2nd order ICIS is found. The collection of the restricted 2nd order ICIS limit sets consist of six distinct points (figure 2.108 (a)). After one iteration we have twelve distinct limit points (figure 2.108 (b)). This process is continued to form a fractal limit set (figure 2.108 (c)).

Example:

Mandelbrot’s Algorithm (Mandelbrot, 1983)

Mandelbrot’s Algorithm begins with a set that we know contains the restricted \( n^{th} \) order ICIS limit set and determines subsets which do not contain any limit points. These subsets of non-limit points are inverted to form further non-limit point subsets hence giving an approximation to the restricted \( n^{th} \) order ICIS limit set.

Example:

Begin with three totally disjoint circles in the plane. We know that the circle inversion limit set applied to three non-overlapping circles is contained within the unique orthogonal circle \((O)\) (Mandelbrot, 1983).
To apply Mandelbrot’s Algorithm we must find arcs of the orthogonal circle which contain no limit points and iterate them to remove other non-limit points. We know that no limit point can lie outside the circles of inversion. The arcs of the orthogonal circle (O) which lie outside the inversion circle will contain non-limit points. These arcs are seen in figure 2.110.

Using Mandelbrot’s Algorithm we say that if \( A_1, A_2, \) and \( A_3 \) contain no limit points then iteration of these arcs will also contain no limit points.

![Inversion of each of the individual \( A_i \)'s](image1)

Figure 2.110

Inversion of each of the individual \( A_i \)'s, where \( i = 1, 2, 3 \), in all circles will remove the following sections of the orthogonal circle:

![Inversion of Arcs](image2)

Figure 2.111
Hence after one application (removal of arcs \( A_1 \), \( A_2 \) and \( A_3 \)) Mandelbrot’s Algorithm produces:

![Figure 2.112](image)

After two applications Mandelbrot’s Algorithm produces:

![Figure 2.113](image)

In effect this method is analogous, topologically speaking, to the construction of The Cantor Middle Third Set.

A Cantor set is a compact, perfect, totally disconnected set (Wachsmuth, 2000). We can use Mandelbrot’s Algorithm to show that the restricted 3\(^{rd}\) order ICIS limit set on a non-overlapping circle placement in which at least two circles are totally disjoint is indeed a Cantor set. This algorithm will give an iterative process which at each step removes a subset of the original set (in this case the orthogonal circle). Like The Cantor Middle Third Set construction at each step a smaller interval/arc is removed, but unlike The Cantor Middle Third Set the length of these arcs are not in the same ratio in subsequent iterations. We begin by removing the arcs, contained within the circumference of orthogonal circle, which lie outside the circles of inversion. Inversion
across all initial circles produce arcs of reduced length. The reduction in length of each arc, at each subsequent iteration, and the removal of these arcs from the remaining portions of orthogonal circle contained in the circles of inversion enabled us to denote the resulting containment set in terms of set exclusion. This process is analogous to that used in the construction of The Cantor Middle Third Set (see appendices). Given the initial set is the orthogonal circle ($\Omega$) then at each iteration ($n$) the containment set has the form:

$$\Omega_n = \Omega_{n-1} \setminus \bigcup \text{open sets}$$

where the resulting limit set $\mathcal{L}(X) = \bigcap \Omega_n$.

**Note:** The arcs $A_1$, $A_2$ and $A_3$ lie outside the circles of inversion. As will be shown in §2 the arcs which lie between known limit points, found via the construction of all restricted 2nd order ICIS limit sets, could also be used. Although Mandelbrot’s Algorithm works by iterating any set of non-limit points, careful choice of the initial set will enhance the effectiveness of the algorithm. Using arcs between known limit points would prove more efficient since we begin by removing larger arcs.

**§2 Classification of Restricted $n^{th}$ order ICIS Limit Sets**

In this section we extend the work of Frame et al (2005) via the classification of limit sets produced under a restricted $n^{th}$ order ICIS. This classification is based on the initial placements of the inverting circles.

**Theorem 2.201 - Classification Theorem for a Restricted $n^{th}$ Order ICIS**

Let $\{C_k\}_{k=1}^n$ be $n \geq 3$ distinct non-overlapping circles contained within the metric space $\hat{\mathbb{R}}^2$. Then the limit set $\mathcal{L}(\{C_k\}_{k=1}^n)$ formed under a restricted $n^{th}$ order ICIS will have one of the following forms:
1. A circle.
2. A fractal dust contained within a circle.
3. A fractal dust contained within multiple circles.
4. A fractal with dimension between one and two which contains a non-fractal subset.

Note:

i. The term fractal dust will be used to describe a Cantor dust produced via circle inversion iteration on $n \geq 3$ non-overlapping circles.

ii. For $n = 3$ only cases one and two apply as discussed in Chapter 2 §1.

iii. For the purpose of this thesis we will view a line as a circle of infinite radius.

iv. The images presented are approximations only. The algorithm used is to create these images was given in Definition 2.102. More detailed images can be produced using software developed by Frame and Booth (2003) which can be found at:

http://classes.yale.edu/Fractals/Software/circlimset.html.

Case 1:
Case 2:

Case 3:
§3 Proof of Classification Theorem

Theorem 2.201 relies on looking at the geometry of the possible placement of \( n \geq 3 \) inversion circles. Our proof relies on the following ideas:

1. **Containment** – For a given set of \( n \geq 3 \) distinct non-overlapping circles the largest convex set that can be formed from the centres of the circles will be called the **Containment Polygon** for the restricted \( n^{th} \) order ICIS.

2. **Orthogonal Circles** – For a given set of \( n \geq 3 \) distinct non-overlapping circles each set of three distinct circles, termed a 3-tuple, can be used to generate an orthogonal circle (Monge’s Problem – see appendix). Given \( n \geq 3 \) distinct non-overlapping circles there exists \( \lambda \) orthogonal circles formed using all possible 3-tuples where \( 1 < \lambda \leq C_3 \).

1. **Containment**:

We will begin with a restricted 3\(^{rd}\) order ICIS and prove the existence of a containment triangle.
Lemma 2.301 - Containment Triangle

The limit set of a restricted 3\textsuperscript{rd} order ICIS is always contained inside a containment triangle $T$ with the centre of the circles as vertices.

![Figure 2.301](image)

Proof:

This proof is based on The Random Inversion Algorithm for a Restricted $n^{th}$ order ICIS. Begin with a point $x_0$ which lies outside all given circles. After one iteration, say in circle $C_1$:

a. The inverse point $x_1$ lies inside circle $C_1$ and inside $T$

or

b. The inverse point $x_1$ lies inside circle $C_1$ but outside $T$.

![Figure 2.302 – Case a](image)
Case a:

Once a point is inside $T$ all subsequent inversion points will also lie within $T$.
The definition of circle inversion (Definition 1.501- Inversion) tells us that all
subsequent inversions in all circles lie on a line segment found by joining our point, $x_n$, and the centre of the circle of inversion. Since $T$ is convex these subsequent inversion points will lie within $T$.

Case b:

Given our inversion point $x_1$ lies outside $T$ further inversion in say $C_2$ will produce one of the following outcomes:

1. The inverse point $x_2$ lies inside circle $C_2$ and inside $T$

   or

2. The inverse point $x_2$ lies inside circle $C_2$ but outside $T$. 
If after two inversions the inversion point, $x_2$, still lies outside $T$ the following options exist:

i. Inversion of $x_2$ in $C_1$ will produce an inversion $x_3$ which lies within $T$.

ii. Inversion alternates between $C_1$ and $C_2$ with an eventual inversion in $C_3$ and hence inversion points contained within $T$.

iii. Inversion alternates between $C_1$ and $C_2$
If inversion alternates between $C_1$ and $C_2$ our inversion points will converge to the boundary of $T$, since the limit set of two circles consists of either one unique or two distinct points which lie on the line segment between the centres of $C_1$ and $C_2$ (Frame et al, 2005).

If the three circle placement has collinear centres then the containment triangle reduces to the line segment joining the centres of the circles of inversion. This family of placements have all limit points contained within a line segment, this segment contained within the line orthogonal to all three circles.

\begin{itemize}
  \item ...
\end{itemize}
For a restricted $n^{th}$ order ICIS we can define our containment polygon in terms of containment triangles.

**Lemma 2.302 - Containment Polygon**

The limit set of a restricted $n^{th}$ order ICIS, $n \geq 3$, is always contained inside the convex containment polygon $Q$. Denote the containment triangles associated with the circles $C_i$, $C_j$ and $C_k$ by $T_{ijk}$, then the containment polygon $Q$ can be formed by taking the union of all containment triangles $T_{ijk}$:

$$Q = \bigcup T_{ijk}.$$  

**Proof:**

$Q$ is union of the $^nC_3$ containment triangles $T_{ijk}$. The containment triangles $T_{ijk}$ are formed by joining the centres of circles $C_i$, $C_j$ and $C_k$. Line segments are formed by joining the centres of all pairs of circles creating a closed polygon containing all triangles. That is, $Q$ is a polygon, an example of which is seen in figure 2.308.

![Figure 2.308 – Five circle placement with 10 containment triangles](image)

If all circles in the placement have collinear centres $Q$ is a line segment. This family of placements have all limit points contained within a line segment; this segment is contained within the line orthogonal to all circles.

We must show that all limit points are contained within or on the boundary of $Q$ and that $Q$ is a convex polygon.

By construction all diagonals of $Q$ lie entirely within $Q$, a defining property of convex polygons, hence $Q$ is a convex polygon.
We must now prove containment. We can always find a limit point by finding the restricted 2nd order ICIS limit set for any pair of distinct circles \( C_i \) and \( C_j \). This limit point will lie on the boundary of containment triangles \( T_{ijk} \) where \( i \neq j \neq k \). By Theorem 2.101 - The Random Iteration Theorem for a Restricted \( n^{th} \) order ICIS a sequence of points can be formed which will eventually fill the limit set densely. Since we know our initial set is contained within multiple containment triangles and since \( Q \) is convex all limit points lie within or on the boundary of \( Q \). That is, under circle inversion mappings \( i_{C_i} \), the image of \( Q \) is contained within the polygon \( Q \).

2. Orthogonal Circles:

We have already determined that the limit set formed under a restricted \( n^{th} \) order ICIS is contained within a polygon. We can further refine this containment using orthogonal circles by first determining the number of orthogonal circles produced using a given \( n \) circle placement and their position in relation to the containment polygon \( Q \).

Given a placement containing \( n \) distinct non-overlapping circles there will exist up to \( \binom{n}{3} \) distinct orthogonal circles. Given three distinct non-overlapping circles \( C_i, C_j \) and \( C_k \) denote the circle orthogonal to \( C_i, C_j \) and \( C_k \) by \( O_{ijk} \).

Given \( n \) distinct non-overlapping circles, if there exists two or more distinct 3-tuples which have a common orthogonal circle we will not have \( \binom{n}{3} \) distinct orthogonal circles. For example, if an \( n \) distinct non-overlapping circle placement has \( n \)-fold rotational symmetry there exists a unique orthogonal circle.

In general to have \( \lambda \) unique orthogonal circles where \( 1 < \lambda \leq \binom{n}{3} \) there must exist at least one subset of \( k \) or more circles, where \( k \neq n \), which themselves have a unique orthogonal circle, as seen in figure 2.309.

Figure 2.309 is a six circle placement which has \( \binom{6}{3} = 20 \) possible 3-tuple orthogonal circles. In this placement the orthogonal circle for the triples
\[
O_{125} = O_{124} = O_{145} = O_{245} = O_4 \quad \text{and} \quad O_{236} = O_{256} = O_{25} = O_{356} = O_{35} = O_8. 
\]
There are only 11 distinct orthogonal circles for this placement.
Given the existence of a containment polygon and orthogonal circles we can now begin to prove Theorem 2.201.

**Proof:**

The four cases presented in Theorem 2.201 can be expressed in terms of the relative placement of the orthogonal circle(s) in relation to the containment polygon $Q$.

1. There exists a unique circle orthogonal to all inversion circles wholly contained within $Q$.
2. There exists a unique circle orthogonal to all inversion circles not wholly contained within $Q$.
3. There does not exist a unique circle orthogonal to all inversion circles and there exists no $k$-tuple, where $3 \leq k < n$, of the form described in case one i.e. no non-fractal (circle) subsets.
4. There does not exist a unique circle orthogonal to all inversion circles and there exists at least one $k$-tuple, where $3 \leq k < n$, of the form described in case one i.e. a non-fractal (circle) subsets.

For a random placement it is most likely to have the form described in case three, because of the non-fractal subset/unique orthogonal circle required in the other three cases.
Case 1:

The $n$ circle placement is pairwise tangent and has a unique orthogonal circle. Mandelbrot (1983) observed that when all the $C_n$'s are orthogonal to a common circle $\Gamma$, the circle $\Gamma$ is self-invariant. The orthogonal circle $\Gamma$ is wholly contained within the containment polygon $Q$. Every circle in the placement is tangent to exactly two other circles and no section of the unique orthogonal circle lies outside the containment polygon $Q$. This results in no non-limit points being removed via Mandelbrot’s Algorithm and hence the entire orthogonal circle $\Gamma$ remains and forms the limit set of the restricted $n^{th}$ order ICIS.

Assume there exists a point contained within the limit set of the restricted $n^{th}$ order ICIS which does not lie on the unique orthogonal circle. This point must lie within the containment polygon (or on the boundary). Inversion of this point will not lie on the unique orthogonal circle but repeated inversions will converge to the unique orthogonal circle due to the convexity of the containment polygon. Hence this point is not a limit point, it does however converge to a limit point contained within the unique orthogonal circle i.e. the limit set of the restricted $n^{th}$ order ICIS.

Case 2:

The $n$ circle placement has a unique orthogonal circle. This circle is not wholly contained within the containment polygon $Q$. Within the placement there exist at least two circles which are totally disjoint. Placements of this form have at least one section/arc of the orthogonal circle which lies outside the containment polygon $Q$, these sections/arcs contain non-limit points and repeated iteration of these non-limit sets will remove arcs from our orthogonal circle (Mandelbrot’s Algorithm). This process is analogous to the construction of the Cantor set and when repeated ad infinitum will produce a fractal dust. This fractal dust is contained within the orthogonal circle and will have a fractal dimension strictly less that one.

If the circle placement has collinear centres the limit set is contained within the line segment which contains the centres of all initial circles. The orthogonal circle is an infinite line and the containment polygon $Q$ is a line segment. Since we can find a line
segment (or ray) on the infinite line which is not contained within the containment line segment we can apply Mandelbrot’s Algorithm to produce a fractal dust.

**Case 3:**

The $n$ circle placement does not have a unique orthogonal circle. The multiple orthogonal circles may or may not be wholly contained within the containment polygon $Q$. No $k$-tuple, where $3 \leq k < n$, has the form described in case one, that is, each $k$-tuple has a unique fractal dust limit set. These $k$-tuple limit sets are subsets of the limit set of the restricted $n^\text{th}$ order ICIS. The construction of the restricted $n^\text{th}$ order ICIS relies on the repeated iteration of these $k$-tuple limit sets. The $k$-tuple limit sets are contained within the arcs of the orthogonal circles as well as the arcs of inverted orthogonal circles, hence on multiple circles. These placements will produce limit sets which are fractal dusts, more specifically the infinite collection of fractal dusts and as such will have a fractal dimension strictly less than one. The fractal dimension is found using Theorem 2.301.

**Theorem 2.301** (Barnsley, 1988)

Given two sets $A$ and $B$. Let $D(A)$ and $D(B)$ represent their fractal dimension and let $D(A \cup B)$ represent the fractal dimension of $A \cup B$. Suppose $D(B) \leq D(A)$. Then

$$D(A \cup B) = D(A).$$

**Case 4:**

The $n$ circle placement does not have a unique orthogonal circle. The multiple orthogonal circles may or may not be wholly contained within the containment polygon. There exists at least one $k$-tuple, where $3 \leq k < n$, which has the form described in case one, that is, there exists a non-fractal subset. Case four is analogous to case three except that a non-fractal limit set is a subset of the eventual limit set. The circle inversion iteration limit set still lies on multiple circles. The repeated iteration of a non-fractal subset will produce further non-fractal subsets, hence the fractal dimension of the overall limit set of the restricted $n^\text{th}$ order ICIS is between one and two.

■
In summary, the classification of a restricted $n^{th}$ order ICIS limit set can be seen in the following diagram:
Chapter 3 Extensions of Circle Inversion Iteration

Many avenues of extension exist in relation to the field of circle inversion iteration. In this thesis we will focus on two, these being iteration of overlapping circles and iteration in three dimensions.

§1 Overlapping Circles and $n^{th}$ order ICISs

Frame et al (2005) have begun work on iterating overlapping circles to produce fractal images. An $n^{th}$ order ICIS can be applied to any circle placement with $\mathbb{R}^2$. In Chapter 2 we imposed the restriction of inversion mappings with scaling mappings $s$ where, $0 \leq s \leq 1$ and only non-overlapping circles placements were considered. This stipulation ensured the formation of nested sets of circles which converge to a unique limit set. These limit sets can be determined on the initial placement of the given circles (Theorem 2.201).

If the interiors of the circles of inversion overlap we can not assume convergence to a limit set. Under a restricted $n^{th}$ order ICIS once a point is contained with a circle of inversion all subsequent inversion must lie within or on the boundary of an initial circle of inversion. When circles of inversion overlap it is possible for a point within a circle of inversion, more specifically in the section of overlap, to be inverted outside all circles of inversion due to Property 1.506 – Involution. When dealing with overlapping circles the domains of our inversion mappings, $i_{C_i}$, are now unrestricted.

Example:

The following example shows inversion performed on two overlapping circles with an initial point of inversion which lies outside both circles $C_1$ and $C_2$. Figure 3.101 was produced using five iterations of Random Inversion Algorithm for a Restricted $n^{th}$ order ICIS (Definition 2.102). It is important to note that the initial condition specified in Definition 2.102 was a non-overlapping initial circle placement, as we will
see the approximations formed using Definition 2.102 are only a guide to what possible limit sets can be formed, if any such limit sets exist at all.

Figure 3.101

Figure 3.102 shows the first 1000 applications of The Random Inversion Algorithm for a Restricted $n^{th}$ order ICIS on four overlapping circles.

Since convergence is not established no limit set may exist. If a limit set does exist it may or may not be fractal.

Example:
Figure 3.103 shows points which extend outside the circles of inversion yet the figure appears to have a fractal nature. From inspection the limit set appears to have non-fractal elements similar to those previously seen in Theorem 2.201 Case 4.

![Figure 3.104]

Figure 3.104 shows a limit set which appears to have a non-fractal nature. This is similar to the limit sets described in Theorem 2.201 Case 1.

From inspection the above figures suggest that fractal limit sets derived from circle inversion iteration are not limited to non-overlapping circle placements. When dealing with overlapping circles Frame et al (2005) propose that the angle of intersection between circles will impose a relation between the inversions.

Frame et al (2005) state:

> „Sometimes, the limit set is more complicated if the circles overlap. If at the point of intersection, the angle between the tangents is a rational multiple of \(\pi\), precise algebraic relations hold between the inversions. These have implications for computing the limit sets“.

**Theorem 3.101 - Overlapping Circles** (Inversion Relations) (Frame et al, 2005)

Distinct circles \(C_1\) and \(C_2\) intersect with tangents making an angle of \(\frac{m}{n} \times 180^\circ\), \(0 < m < n\), if and only if the corresponding inversion \(I_1\) and \(I_2\) satisfy \((I_1I_2)^n = \text{identity}\).
Proof:

Suppose $C_1$ and $C_2$ intersect at two points $p$ and $q$.

![Figure 3.105](image)

Construct a circle $C$ with centre $p$ and which passes through $q$.

![Figure 3.106](image)

Invert $C_1$ in $C$ to produce line $L_1$ and invert $C_2$ in $C$ to produce line $L_2$ (Property 1.501 (b)).

![Figure 3.107](image)
Inversion in $C$ transforms the inversions in $C_1$ (i.e. $I_1$) and $C_2$ (i.e. $I_2$) into the reflections $R_1$ across $L_1$ and $R_2$ across $L_2$. Property 1.508 states that inversion is a conformal mapping i.e. angle preserving, hence the angle between $L_1$ and $L_2$ at the point $q$ is equal to the angle between the tangents of $C_1$ and $C_2$ at $q$.

Since conjugation by the inversion in $C$ transforms $I_1$ and $I_2$ into the reflections then the composition of the two reflections across lines intersecting at the point $q$ is equivalent to the rotation, about $q$, through twice the angle between the lines $L_1$ and $L_2$.

Denote the angle between $L_1$ and $L_2$ as $\theta$.

If the angle between $L_1$ and $L_2$ is $\frac{m}{n} \times 180^\circ$, then

\[
(R_1R_2)^n = 2n \times \frac{m}{n} \times 180^\circ \\
= m \times 360^\circ \\
= \text{identity}
\]

Using inversion in $C$ to translate this relation back to $I_1$ and $I_2$, we see that $(I_1I_2)^n = \text{identity}$.

Conversely,

\[
R_1R_2 = \text{rotation by } 2\theta \text{ at point } q
\]

and

\[
(I_1I_2)^n = \text{identity} \\
\Rightarrow n \times 2\theta = m \times 360^\circ \\
\therefore \theta = \frac{m}{n} \times 180^\circ
\]

If $C_1$ and $C_2$ intersect tangentially at point $p$ the above process will produce parallel lines $L_1$ and $L_2$ (Property 1.501 (b) – Inversion of a Circle) and hence the relation $(R_1R_2)^n = \text{identity}$ does not hold. For the relation $(I_1I_2)^n = \text{identity}$ to hold circles $C_1$ and $C_2$ must intersect at two distinct points $p$ and $q$. 

■
This relation relates overlapping circles to the disjoint circle relation which states “If the open discs $D_i$ bounded by the inverting circles $C_i$ are pairwise disjoint, the only relations among the inversions are $I_i^2 = \text{identity}$” (Frame et al, 2005).

Although the above theorem defines an algebraic relation on overlapping circles it does not detail the impact this relation has on inversion limit sets. There are infinitely many combinations of $m$ and $n$ that will hold yet vastly different limit sets seem possible with only a small variance in the angle of intersections.

In their research Frame et al (2005) give examples of overlapping circles with intersections of 60° and 90°.

Example:

![Figure 3.108 – Eight circles with non-tangential intersection of 90° (Frame and Booth, 2003)](image1)

Figure 3.108 (ii) has a fractal structure with fractal dimension between one and two, the limit set is wholly contained within the unique orthogonal circle and it exhibits properties similar to those found in Theorem 2.201 Case 4. Figure 3.108 (i) shows the angles of intersection of the inversion circles is equal to 90°. Even when angles of intersection are of the form described in Theorem 3.101 containment is not ensured, as shown in figure 3.109.
The elements of the limit set which lie outside the initial inversion circles are formed by inverting a point inside a given circle to a point outside the given circles, a consequence of overlapping circles. In order to combat this a restricted limit set can be formed.

Clancy and Frame (1996) term the restricted limit set as the “limit set of the orbit of a point, with the restriction that if some orbit point \( x_i \) lies in the disc bounded by \( C_j \), then the next orbit point \( x_{i+1} \), cannot be in \( I_j(x_i) \)” . If the circle placement has disjoint interiors the above restriction reduces to “never inverting in the same circle twice”. Furthermore the restricted limit set will be contained within, or on the boundary of, the circles \( C_i \). If there exists a region common to all circles in the placement an initial set outside this common region must be chosen in order to generate the limit set. Figure 3.110 shows the restricted limit set for the circle placement shown in figure 3.109.
This restricted limit set in relation to the inversion limit set can be clearly seen in figure 3.111.

From the work of Frame et al (2005) and Frame and Booth (2003) it is clear that fractal inversion limit sets exist for overlapping circle placements. Unlike the work undertaken for non-overlapping circles strict criteria has not yet been established to predict the eventual structure of the inversion limit set. Since under inversion applied to overlapping circles the inversion mappings may not be contraction mappings, as the domains of the mappings are unrestricted, convergence to a limit set is not ensured. Since it appears that some circle placements have unbounded limit sets the use of restricted limit sets may aid in creating detailed limit sets which display fractal properties.

§2 Spherical Inversion

In Chapter 2 we explored how circle inversion iteration can produce fractal limit sets. Circle inversion as we currently view it is confined to the extended real plane $\hat{\mathbb{R}}^2$. One possible avenue for expansion is to apply the concepts and properties of circle inversion to spheres and planes within $\hat{\mathbb{R}}^3$. Inversion in three dimensions, commonly termed spherical inversion, is seen by many authors (Leys, 2009) as being equivalent to circle inversion just in three dimensions. Leys (2009) states “sphere inversion transformations are the 3D equivalent of circle inversions”. At present the study of
spherical inversion is limited. Authors such as Leys (2009) have however created spherical inversion limit set images via computer software, these limit sets displaying fractal properties. We will formally define spherical inversion and review its properties. Once defined we will relate spherical inversion to iterative mapping systems by defining an ISIS (Iterated Spherical Inversion System). Once defined we will place restrictions on the system in order to produce fractal limit sets.

**Definition 3.201 - Spherical Inversion** \( (i_S) \) (Kelly, 1970)

Given a fixed sphere of radius \( k \) and centre \( O \) we define the inverse of any point \( P \) (distinct from \( O \)) to be the point \( P' \) on the ray \( OP \) whose distance from \( O \) satisfies the equation

\[
OP \times OP' = k^2
\]

**Note:**

i. \( i_S \) defines inversion across sphere \( S_k \).

ii. We will view a plane as a sphere of infinite radius.

iii. We will view a circle as the intersection of two spheres.

Properties of circle inversion are seen in spherical inversion. We will view circle inversion as a subset of spherical inversion. It can be proved that any sphere inverts into another sphere.
**Theorem 3.201** (Kelly, 1970)

Given a sphere $S$ with centre $O$ and radius $k$, called the sphere of inversion, the inverse of sphere $\sigma$ with centre $C$ and radius $r_1$ across $S$ is another sphere $\sigma'$ centre $D$. Hence $\sigma$ is the inverse of $\sigma'$ in $S$ and visa versa.

![Diagram](i)(ii) Figure 3.302 (i)(Joyce, 2002) and (ii) (Leys, 2008)

**Corollary 3.201**

Since any circle can be regarded as the intersection of two spheres, we have also shown that all circles invert to circles.

**Corollary 3.202**

If a sphere $\sigma$ passes through $O$, $\sigma'$ is a plane.

**Corollary 3.203**

The centre of sphere $\sigma$, $C$, is in general not the inverse of the centre of sphere $\sigma'$, $D$. The centre of sphere $\sigma$, $C$, is the inverse of the centre of sphere $\sigma'$, $D$ if and only if the spheres $S$ and $\sigma'$ share a common centre i.e. are concentric.

**Spherical Inversion Formulas:**

Frame et al (2005) show that given a circle $C$, centre $P = (a, b)$ radius $r$, we know that the point $(x, y)$ has a corresponding inverse point $P' = (x', y')$ which is given by:
Frame et al (2005) details the formula required to find the inverse of a given circle, its centre and radius. In order to find the equation of the newly inverted sphere

\[(x - i)^2 + (y - j)^2 + (z - k)^2 = r^2\]

we need to solve for four variables. To accomplish this we must first find a plane passing through both the centre of the sphere of inversion, \(S\), and our initial sphere, \(\sigma\). Within this plane our two spheres are reduced to circles and we can apply the work of Frame et al (2005) to find the circle of inversion.

We can determine the equation of the inverted sphere a number of ways:

a) Using the work of Frame et al (2005) in relation to circle inversion we can reduce our spheres to two circle within a plane and find:

i. The centre of the newly inverted circle and hence the centre of newly inverted sphere.

ii. The radius of the newly inverted circle and hence the radius of newly inverted sphere.

iii. Given the centre and radius of the newly inverted circle we can determine its equation and hence find multiple points on its circumference. All point on the circumference of the newly inverted circle will lie on the boundary of the newly inverted sphere.

By finding three points distinct points which lie on the circumference of the newly inverted circle/sphere as well as determining the spherical radius we can uniquely define a sphere. This enables us to determine the equation of the newly inverted sphere.

Alternatively,
b) By choosing a plane of intersection passing through the centres of the spheres of inversion we will have infinitely many points of contact between the initial sphere and the plane i.e. all points on the circumference of the common circle. By choosing four distinct points on the circumference and inverting them we create four distinct inversion points which are contained within the circumference of the newly inverted circle and hence the newly inverted sphere. Solving a system of linear equations will determine the equation of the newly inverted sphere.

The following properties exist in relation to spherical inversion:

Inversion across a sphere $S$ with centre $o$

1. leaves unchanged every point on the sphere $S$.
2. interchanges the inside and outside of the sphere $S$.
3. is a contraction on sets outside the sphere $S$.

**Note:** Inversion across a sphere $S$ with centre $o$ is only a contraction on sets exterior to the sphere of inversion.

4. preserves spheres orthogonal to the inverting sphere $S$.
5. is an involution.
6. takes a sphere not passing through $o$ to a sphere not passing through $o$.
7. takes a sphere passing through $o$ to a plane.

As with circle inversion, spherical inversion uses the property of orthogonality.

**Definition 3.202 - Orthogonal Spheres** (Boehm and Prautzsh, 1994)

There exists exactly one sphere orthogonal to four spheres in $\mathbb{R}^3$.

We can define an Iterated Spherical Inversion System (ISIS) which we can use to create limit sets, some of which are fractal in nature.
§3 Iterated Spherical Inversion Systems (ISISs)/Restricted ISISs

**Definition 3.301 - Iterated Spherical Inversion System (ISIS) / \( n^{th} \) order ISIS**

An Iterated Spherical Inversion System (ISIS) is a finite collection \( I = \left\{ i_{S_k} \right\}_{k=1}^{n} \) of spherical inversion mappings \( i_{S_k} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), where the \( S_k \)'s are distinct.

As with ICISs ISISs do not satisfy the criteria of IFSs, as such we can not prove using the mechanics of IFSs that a unique limit set exists. We can however show a restricted \( n^{th} \) order ISIS convergences to a unique limit set in a manner analogous to that described for a restricted \( n^{th} \) order ICIS. As seem in relation to restricted \( n^{th} \) order ICISs the restrictions imposed on a restricted \( n^{th} \) order ISIS will be that the mappings \( i_{S_k} \) must be defined on a restricted domain such that the scaling factor \( s \) of all mappings satisfies \( 0 \leq s \leq 1 \) and all spheres of inversion are non-overlapping i.e. spheres that have disjoint interiors and at most one point of contact between the boundaries.

In order to aid the visualisation of the limit sets for a restricted \( n^{th} \) order ISIS we can first determine the inversion iteration limit sets formed using two and three non-overlapping spheres i.e. the restricted 2\(^{nd}\) and 3\(^{rd}\) order ISIS limit sets.

**Example:**

We know that a sphere inverts to another sphere (Theorem 3.201) and repeated inversion of these spheres will produce a nested set of spheres with the diameter at each iteration decreasing i.e. throughout the inversion process the length of the diameter is converging to zero. If \( S_A \) and \( S_B \) are totally disjoint then the limit set consists of two distinct points. If \( S_A \) and \( S_B \) have a point of intersection the spherical inversion limit set consists of a single point, the point of tangency.

Spherical inversion iteration using only two non-overlapping spheres render the same limit set as for two non-overlapping circles. This is due to the fact that there exists infinitely many planes containing the centres of the inverting spheres \( S_A \) and \( S_B \). Each of these planes cut the spheres to produce two non-overlapping circles in a plane. Hence
for all possible planes the limit set formed under circle inversion iteration is fixed and hence this fixed set holds for spherical inversion.

When dealing with three non-overlapping spheres we can not find a unique orthogonal sphere. A minimum of four spheres are needed to define a unique orthogonal sphere (Definition 3.202). Given the centres of the three spheres of inversion there exists a unique plane containing all three points. This plane intersects our inversion spheres, the image of which is equivalent to three non-overlapping circles. These circles have a unique orthogonal circle yet this orthogonal circle can fall on infinitely many spheres orthogonal to the initial spheres of inversion. Using a three dimensional version of Frame’s Algorithm we can deduce that the limit set of a restricted 3rd order ISIS is equivalent to that of a restricted 3rd order ICIS.

Example:

Beginning with the known restricted 2nd order ISIS limit sets we have between three and six known limit points. These limit points are contained in the fixed plane and within the fixed orthogonal circle. Each of these known limit points invert to other limit points, all of which are contained within the unique orthogonal circle. Hence, the limit set of a restricted 3rd order ISIS is equivalent to that of a restricted 3rd order ICIS. The limit set of a restricted 3rd order ISIS is either the unique orthogonal circle or a fractal dust contained within the unique orthogonal circle.

When studying restricted n-th order ICIS limit sets we found that the limit sets can consist of points, circles (closed 2D curves) or a combination of both. When studying restricted n-th order ISIS limit sets we will encounter both fractal and non-fractal limit sets yet limit sets will now incorporate spheres.

The following are examples of the work of Leys (2009). Leys (2009) states:

“Well-chosen initial spheres are iteratively inverted in well-chosen 'inversion spheres' to obtain the patterns in the images below”

At present no classification theorem exists which details the structure spherical inversion iteration limit sets. We have however conjectured on the structure of such limit sets in relation to the initially placed spheres, this conjecture can be found in Chapter 6 §1.
Chapter 4 Iteration of Möbius Transformations

The iterative systems considered thus far have been confined to the metric spaces \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \). In this chapter we will explore iterative systems whose maps operate in the extended complex plane \( \hat{\mathbb{C}} \). The text *Indra’s Pearls – The Vision of Felix Klein* (2002) is a definitive text in this area through which Mumford, Series and Wright investigated limit sets formed under iteration of systems of Möbius transformations. In particular, this review will focus on the iteration of circle pairing Möbius transformations.

In Chapter 1 §6 circle pairing Möbius transformations were discussed (see figure 1.607). Even though Mumford et al (2002) did not view these mappings in terms of an iterative system we can define an iterative system comprising of circle pairing Möbius transformations analogous to that defined for circle inversion mappings. These circle pairing systems will be iterated ad infinitum to create limit sets which may or may not exhibit fractal properties.

**Iterated Circle Pairing System (ICPS)/ \( n^{th} \) order Iterated Circle Pairing System (\( n^{th} \) order ICPS)**

An Iterated Circle Pairing System (ICPS) is a finite collection of circle pairing Möbius transformations \( \{a_i, A_i\}_{i=1}^n \), where \( A_i = a_i^{-1} \). An \( n^{th} \) order ICPS \( (n \geq 1) \) is an ICPS applied to a set of \( 2n \) distinct non-overlapping circles \( C_{a_i} \) and \( C_{A_i} \).

If we denote the interior of the circles \( C_{a_i} \) and \( C_{A_i} \) where \( i = 1,2,3,\ldots,n \) by \( D_{a_i} \) and \( D_{A_i} \) then the maps in the \( n^{th} \) order ICPS are:

\[
\begin{align*}
a_i : \text{exterior of } D_{A_i} & \mapsto \text{interior of } D_{a_i} \\
A_i : \text{interior of } D_{a_i} & \mapsto \text{exterior of } D_{A_i}
\end{align*}
\]

where \( A_i = a_i^{-1} \) and \( i = 1,2,3,\ldots,n \)

Note that \( a_i \) takes the circle \( C_{A_i} \) and maps it to circle \( C_{a_i} \), and similarly for map \( A_i \). As seen in Chapters 2 and 3 two circles will be considered non-overlapping is they have
disjoint interiors and at most one points of contact i.e. a point of tangency. We will not consider ICPSs applied to overlapping circles within this review.

For all of the iterative systems discussed in this review the region, \( R_0 \), which lies outside \( D_{a_i} \) and \( D_{A_i} \) will be taken as the initial set. Iteration of a circle pairing system proceeds as follows:

1. Begin with an initial set \( R_0 \) in \( \hat{C} \).
2. Apply each map to that part of \( R_0 \) which is in its domain. Let \( R_1 \) be the union of all such images.
3. Apply each map to that part of \( R_1 \) which is in its domain. Let \( R_2 \) be the union of all such images.
4. Repeat for \( R_3, R_4, \ldots \)

We are interested in the limit set of this sequence. In particular we are interested in limit sets that are fractal. Note that under the circle pairing mappings the image of \( R_0 \) under \( a_i \) will be in \( D_{a_i} \) and under \( A_i \) will be in \( D_{A_i} \). Thus, after the first iteration we need only deal with our initial disks.

§1 1st order Iterated Circle Pairing Systems (1st order ICPSs)

A 1st order ICPS will consist of one pair of mappings \( a_i \) and \( A_i \) which pair two distinct non-overlapping circles \( C_{a_i} \) and \( C_{A_i} \). These circles bound disks \( D_{a_i} \) and \( D_{A_i} \).

Initially let \( a_i \) and \( A_i \) be maps that pair totally disjoint circles \( C_{a_i} \) and \( C_{A_i} \) as shown in figure 4.101 and let \( R_0 \) be the initial set outside both initial disks.

![Figure 4.101 - 0th iteration](image-url)
Since the circles in figure 4.101 are totally disjoint any map that pairs them must be loxodromic (possibly hyperbolic). As discussed in Chapter 1 there exist an infinite number of circle pairing maps for circles \( C_{a_1} \) and \( C_{A_i} \). Figure 4.102 shows the first iteration of the 1st order ICPS for one possible choice of \( a_i \) and \( A_i \).

Mumford et al (2002) introduced terminology to aid in the study of the iterative process. Mapping \( a_i \) applied to disk \( D_{a_i} \) is denoted by \( a_i(D_{a_i}) \) which will be shortened to \( D_{a_i,a_i} \) or even just the ‘word’ \( a_i a_i \). Since \( a_i \) maps the outside of \( D_{a_i} \) to the inside of \( D_{a_i,A_i} \), \( D_{a_i,a_i} \subset D_{a_i} \). We will use the notation developed by Mumford et al (2002) in our review of ICPSs. The second iteration of the 1st order ICPS is shown in figure 4.103.

This process is continued resulting in a sequence of nested disks e.g. \( D_{a_i} \supset D_{a_i,a_i} \supset D_{a_i,a_i,a_i} \supset \ldots \). This sequence of nested disks converges to the limit set of the 1st order ICPS. The limit set will consist of the fixed points of the mappings, as shown in figure 4.104. Given the two fixed points found under a 1st order ICPS on totally disjoint circles one will be the attracting fixed point \( \text{Fix}^+(a_i) \) of \( a_i \) \( (\text{Fix}^+(a_i) \in D_{a_i}) \) and the other the repelling fixed point \( \text{Fix}^-(a_i) \) of \( a_i \) \( (\text{Fix}^-(a_i) \in D_{A_i}) \).
Thus for any pair of disjoint circles iteration of any circle pairing mapping will result in a limit set consisting of just two points, the fixed points. Although the pairing maps \( a_i \) and \( A_i \) are not unique to the placement if the circles \( C_{a_i} \) and \( C_{A_i} \) are disjoint the limit set will always be two distinct points.

Constructing an ICPS limit set in the manner described above is called a breadth-first algorithm. This is to distinguish it from another approach called the depth-first algorithm. The depth-first algorithm follows a sequence of maps until a limit point is reached, the algorithm then returns to the original placement and begins a new sequence, this is continued until all limit points are formed. In the case of a 1st order ICPS only two sequences can be formed \( a_1 a_1 a_1 a_1 \ldots \) and \( A_1 A_1 A_1 A_1 \ldots \). These infinite words correspond to the fixed points of the two sequences. Note that since \( a_i A_i = I = A_i a_i \), the maps \( A_i(D_{a_i}) \) and \( a_i(D_{A_i}) \) are excluded from the construction of a 1st order ICPS.

When dealing with tangent circles the 1st order ICPS limit set may be a single point or a limiting circle containing two fixed points. If the map \( a_i \) pairing the two circles is parabolic then the point of tangency is fixed under iterations of \( a_i \), that is, the point of tangency is a fixed point where \( \text{Fix}^+ (a_i) = \text{Fix}^- (a_i) \). In this case the limit set of a 1st order ICPS will consist of a single point. If the map \( a_i \) pairing the two tangent circles is loxodromic (or hyperbolic) then two fixed points exist, both of which lie within or on the boundary of one disk, an example of which is shown in figure 4.105. In figure 4.105 iteration of map \( a_i \) on disk \( D_{a_i} \) will produce a sequence of disks nesting down on \( \text{Fix}^+ (a_i) \in D_{a_i} \) where the point of tangency equals \( \text{Fix}^+ (a_i) \). Iteration of map

![Diagram](image-url)
A_i on disk \( D_{\alpha_i} \) will produce a sequence of disks which at each iteration must contain both fixed points. This sequence converges to a curve (circle) which contains the two distinct fixed points on its boundary. That is, the resulting image of the disks under the mappings does not nest down to \( \text{Fix}^{-1}(a_i) \subseteq D_{\alpha_i} \).

![Figure 4.105 – Loxodromic (possibly hyperbolic) circle pairing mapping (Mumford et al, 2002)](image)

§2 2nd order Iterated Circle Pairing Systems (2nd order ICPSs)

As seen in §1 the limit set of a 1st order ICPS is never fractal. However, fractal limit sets can be produced under a 2nd order ICPS. A 2nd order ICPS limit set will have one of the following structures:

1. A fractal dust.
2. A non-fractal (circle/line).
3. A fractal with fractal dimension between one and two.

Conditions/criteria required to form the above mention limit set structures exist and will be discussed in this section. It is interesting to note that limit sets with the above structures can be created via iteration of various restricted \( n \text{th} \) order ICISs. The limit set formed under a restricted \( n \text{th} \) order ICIS is completely determined by the placement of the initial circles and its structure predicted using only the placement of the initial circles. This is not true however for limit sets formed under a 2nd order ICPS, while the structure depends on the placement of the initial circles it is not completely determined by that placement.
Example: (Fractal Dust)

Given disks $D_{a_1}, D_{A_1}, D_{a_2}$ and $D_{A_2}$ shown in figure 4.201 and the mappings $a_1$ and $a_2$ where

$$a_1 = \begin{pmatrix} 4/9 & 7/9 \\ 1/9 & 4/9 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & -5 \\ -1/4 & 1 \end{pmatrix}$$

a fractal limit set is formed.

![Figure 4.201](image)

Note that the circle pairing mappings used in this chapter were found using the formula described in Chapter 1 §6 page 35. The matrices given are in normalized form.

A 2nd order ICPS on disks $D_{a_1}, D_{A_1}, D_{a_2}$ and $D_{A_2}$ uses four mappings $a_1, a_2, A_1$ and $A_2$ where $A_i = a_i^{-1}$ for $i=1, 2$. Thinking in terms of the depth-first algorithm under a 2nd order ICPS our ‘alphabet’ has increase from two letters to four allowing more words to be formed. One such example is seen using the word $D_{a_1 A_1 A_1 A_1 A_1}$ where

$$D_{a_1 A_1 A_1 A_1 A_1} \subset D_{a_1 A_1 A_1 A_1 A_1} \subset D_{a_1 A_1 A_1 A_1 A_1} \subset D_{a_1 A_1 A_1 A_1 A_1} \subset D_{a_1 A_1 A_1 A_1 A_1} \subset D_{a_1 A_1 A_1 A_1 A_1}.$$ This will result in our limit set consisting of more than two points as we can now form an infinite number of words resulting in an infinite number of limit points. As previously discussed all combination of letters $a_i A_i$ and $A_i a_i$ are excluded since $A_i = a_i^{-1}$. 

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Example: (Fractal Dust)

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$$a_1 = \begin{pmatrix} 4/9 & 7/9 \\ 1/9 & 4/9 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & -5 \\ -1/4 & 1 \end{pmatrix}$$

a fractal limit set is formed.

![Figure 4.201](image)

Note that the circle pairing mappings used in this chapter were found using the formula described in Chapter 1 §6 page 35. The matrices given are in normalized form.

A 2nd order ICPS on disks $D_{a_1}, D_{A_1}, D_{a_2}$ and $D_{A_2}$ uses four mappings $a_1, a_2, A_1$ and $A_2$ where $A_i = a_i^{-1}$ for $i=1, 2$. Thinking in terms of the depth-first algorithm under a 2nd order ICPS our ‘alphabet’ has increase from two letters to four allowing more words to be formed. One such example is seen using the word $D_{a_1 A_1 A_1 A_1 A_1}$ where

$$D_{a_1 A_1 A_1 A_1 A_1} \subset D_{a_1 A_1 A_1 A_1 A_1} \subset D_{a_1 A_1 A_1 A_1 A_1} \subset D_{a_1 A_1 A_1 A_1 A_1} \subset D_{a_1 A_1 A_1 A_1 A_1} \subset D_{a_1 A_1 A_1 A_1 A_1}.$$ This will result in our limit set consisting of more than two points as we can now form an infinite number of words resulting in an infinite number of limit points. As previously discussed all combination of letters $a_i A_i$ and $A_i a_i$ are excluded since $A_i = a_i^{-1}$.
The nested sets of circles associated with the circle placement given in figure 4.201 can be seen in figure 4.202. The nested sets of circles converge to the limit set. The words associated with the first iteration of our 2nd order ICPS are shown.

Figure 4.202 - Array after 8 iterations

Using the above example if we change the order in which the initial circles are paired and define a new system of circle pairing mappings we can create a new limit set using the same initial circle placement.

Example:

The mappings used to create the limit set in figure 4.203(ii) are as follows:

Figure 4.203 – (i) previously seen in figure 4.202
\[
\begin{pmatrix}
-\frac{2}{3} & 1 + \frac{8}{3}i \\
-\frac{1}{6} & -\frac{2}{3}
\end{pmatrix}
\quad
\begin{pmatrix}
\frac{2}{3}i & -1 + \frac{8}{3}i \\
-\frac{1}{6} & -\frac{2}{3}
\end{pmatrix}
\]

In this case the structure of the limit set is still the same i.e. a fractal dust.

The limit set of a 2\textsuperscript{nd} order ICPS will always exhibit a fractal dust structure when the initial circle placement is either totally disjoint or at least two initial circles are totally disjoint. This is due to the disconnection in the original placement which is reflected in subsequent iterations forming a fragmented limit set.

Mumford et al (2002) notes that the collection of all possible compositions of the circle pairing mappings \( a_1 \) and \( a_2 \) as well as their inverses \( A_1 \) and \( A_2 \) form a group. The collection of all compositions (in any order) is called the \textbf{Schottky group on the generators} \( a_1 \) and \( a_2 \). Using the terminology of ICPSs a Schottky group on generator \( a_1 \) and \( a_2 \) correspond to a particular 2\textsuperscript{nd} order ICPS. As such the limit set associated with a Schottky group on generator \( a_1 \) and \( a_2 \), termed a \textbf{Schottky limit set} by Mumford et al (2002), will correspond to the limit set of a particular 2\textsuperscript{nd} order ICPS. From now on we will use the terminology of Mumford et al (2002) and that of ICPSs interchangeably.

We have seen that a totally disjoint placement will produce, under a 2\textsuperscript{nd} order ICPS, a fractal dust limit set. To see cases where the structure of the limit set is not determined by the initial circle placement we must look at placement were the four initial circles form a tangent chain. In order to distinguish these new groups from the Schottky groups made by pairing disjoint circles Mumford et al (2002) termed these new groups of paired tangent circles \textbf{Kissing Schottky groups}. When the circle placement forms a tangent chain the associated 2\textsuperscript{nd} order ICPS limit set can exhibit any of the three previously mentioned structures, that is, such tangent placements can produce limit sets which are non-fractal (circle), fractal dusts or fractals with fractal dimension between one and two.

The following is an example of a tangent chain of four identical circles. In this placement opposite circles have been paired together. This placement and its associated generators produce a fractal dust limit set.
Example:

Given four distinct tangent circles shown in figure 4.204 and the mappings $a_1$ and $a_2$

\[
\begin{pmatrix}
\frac{3}{-2(1+i)} & \frac{-1+i}{-2(1+i)} \\
\frac{1}{-2(1+i)} & \frac{1}{2}
\end{pmatrix}
\]

the limit set shown in figure 4.204 is formed.

\[
\begin{pmatrix}
\frac{3}{-2(1+i)} & \frac{1}{2} \\
\frac{1}{-2(1+i)} & \frac{-1+i}{-2(1+i)}
\end{pmatrix}
\]

The above example failed to produce a connected limit set since the image circles do not nest down at the points of tangency between disks $D_{a_1}$ and $D_{a_2}$ or $D_{a_1}$ and $D_{a_2}$. This results is a disconnection in the image circles and a disconnected limit set, as shown in figure 4.205.
In order to generate some continuous/connected fractal limit sets as well as non-fractal limit sets Mumford et al (2002) focused their study on pairing circles in a tangent chain with imposed conditions on the mappings. They were able to produce conditions which ensure the nested circles resemble ‘beads of a necklace’. These conditions are defined in The Necklace Theorem – Theorem 4.201.

**Theorem 4.201 – The Necklace Theorem** (Mumford et al, 2002)

Four circles $C_{a_1}, C_{a_2}, C_{a_3}$ and $C_{a_4}$ bound disks $D_{a_1}, D_{a_2}, D_{a_3}$ and $D_{a_4}$ where the $C_{a_i}$’s form a tangent chain. The following conditions ensure a group is formed whose limit set is a curve found by pairing opposite disks using two transformations $a_1$ and $a_2$.

1. the tangency points $P, Q, R$ and $S$ must be correctly matched by the generators, as shown in figure 4.206.
2. the trace of $a_1a_2A_1A_2$, denoted by $tr(a_1a_2A_1A_2)$, must be $-2$ i.e. the word $a_1a_2A_1A_2$ is parabolic.

$$
C_{a_1} \cap C_{a_2} = R \\
C_{a_1} \cap C_{a_3} = S \\
C_{a_1} \cap C_{a_4} = P \\
C_{a_2} \cap C_{a_3} = Q \\
C_{a_1} : \text{exterior } C_{a_4} \leftrightarrow \text{interior } C_{a_1} \\
C_{a_2} : \text{exterior } C_{a_4} \leftrightarrow \text{interior } C_{a_2}
$$

![Figure 4.206](image)

The necessary (first) condition for the limit set to be a curve is that the tangent points be mapped as follows:
As a result of this condition after the first iteration within each initial disk there will exist three newly formed circles. In order to form a connected limit set these three circles must also be tangent, as shown in figure 4.208.

Without loss of generality we will focus on the disk $D_{a_1}$. Due to the tangency points we have the following:
Circle $C_{a_1}$ is tangent to circle $C_{a_2}$ and $C_{a_2}$ at points $P$ and $Q$ respectively. The image of disk $D_{a_1}$ under map $a_1$ is a disk $D_{a_1 a_1}$ contained within $D_{a_1}$. Disk $D_{a_1 a_1}$ must contain the image points of $P$ and $Q$ on its boundary since they lie on $C_{a_1}$. Furthermore, the image of disk $D_{a_2}$ under $a_1$ must also contain the image of point $P$ on its boundary and the image of disk $D_{a_2}$ under $a_1$ must also contain the image of point $Q$ on its boundary. Under $a_1$, the points $P$ and $Q$ can only be mapped to one other point hence the image of disks $D_{a_1}$, $D_{a_2}$ and $D_{a_2}$ under mapping $a_1$ form a tangent chain within $D_{a_1}$.

Likewise for the other disks in the placement. These conditions ensure a tangent chain of disks is formed, we can see this tangency is continued throughout the iterations forming a nested chain of disks. Figure 4.210 shows the second iteration of the disks.
Note that as a result of this condition \( a_2(A_2(P)) = P \), that is, \( P \) is a fixed point for the transformation \( a_1a_2A_1A_2 \).

Matching tangent points is not all that is required to create a connected limit set, a second requirement is that the nested circles shrink down to points. Thus the tangency point \( P \) between the circles \( C_{a_1} \) and \( C_{a_2} \) must be the parabolic fixed point of the transformation \( a_1a_2A_1A_2 \). If the transformation word \( a_1a_2A_1A_2 \) is loxodromic then by definition two fixed points exist, these being \( P \) and \( P' \). The existence of a second fixed point may prevent the circles from shrinking to the point \( P \), an example of which was shown in figure 4.105. Hence \( a_1a_2A_1A_2 \) must be parabolic in order to form a continuous limit curve/set.

The word \( a_1a_2A_1A_2 \) is the commutator of the mappings \( a_1 \) and \( a_2 \). We initially chose to work with tangency point \( P \). There exist parabolic commutators for tangency points \( R, S \) and \( Q \). As such we could have started with any tangency point. The associated commutators for all four tangency points are as follows:

\[
\begin{align*}
a_1a_2A_1A_2(P) &= P = a_2a_1A_2A_1(P) \\
A_1A_2a_1a_2(R) &= R = A_2A_1a_2a_1(R) \\
a_1A_2A_2a_1(S) &= S = A_2a_1a_2A_1(S) \\
A_2a_1a_2A_1(Q) &= Q = a_1A_2A_1a_2(Q)
\end{align*}
\]

Since the transformations \( a_1a_2A_1A_2, a_2A_1a_2a_1, A_1a_1a_2A_1 \) and \( A_2a_1a_2A_1 \) are cyclic permutations of each other their traces will be equal i.e. \( tr(a_1a_2A_1A_2) = tr(a_2A_1a_2a_1) = tr(A_1a_1a_2A_1) = tr(A_2a_1a_2A_1) \) and likewise for their inverses \( a_2a_1A_2A_1, A_2a_1a_2A_1, A_1a_2a_1A_2 \) and \( A_2a_1A_2a_1 \).

Since \( a_1a_2A_1A_2 \) is parabolic its trace is either \( \pm 2 \) (assuming determinant 1). The final condition is that \( tr(a_1a_2A_1A_2) \) must always equal \(-2\). If \( tr(a_1a_2A_1A_2) = 2, a_1(P) = P \) and \( a_2(P) = P \) (Jørgensen and Klein, 1982). Since we are pairing opposite circles the mappings \( a_1 \) and \( a_2 \) can not be parabolic. Iteration of maps \( a_1 \) and \( A_1 \) on circles \( C_{a_1} \) and \( C_{a_2} \) produce two distinct fixed points hence that map \( a_1 \) can be represented by its canonical form \( a_1(z) = kz \) for some \( k \) (Chapter 1 §6 pg 32). Assume mapping \( a_2 \) has the
general form of a Möbius transformation i.e. \( a_z(z) = \frac{az + b}{cz + d} \) where \( ad - bc = 1 \). We have the following four mappings:

\[
\begin{align*}
    a_1 &= \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}, \\
    A_1 &= \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{pmatrix}, \\
    & \quad \text{where} \\
    a_2 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \\
    A_2 &= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\end{align*}
\]

where

\[
 a_1a_2A_1A_2 = \begin{pmatrix} ad - kbc & kab - ab \\ cd - kcd & ad - \frac{bc}{k} \end{pmatrix}
\]

\[
\text{tr}(a_1a_2A_1A_2) = 2ad - kbc - \frac{bc}{k}
\]

given \( ad - bc = 1 \Leftrightarrow ad = 1 + bc \)

\[
\text{tr}(a_1a_2A_1A_2) = 2(1 + bc) - kbc - \frac{bc}{k}
\]

\[
= 2 + 2bc - kbc - \frac{bc}{k}
\]

\[
= 2 + bc(2 - k - \frac{1}{k})
\]

\[
= 2 - \frac{bc}{k}(k^2 - 2k + 1)
\]

\[
= 2 - \frac{bc}{k}(k - 1)^2
\]

\[
\text{tr}(a_1a_2A_1A_2) = 2 \Rightarrow 0 = -\frac{bc(k - 1)^2}{k}
\]

In this case we have three possibilities
1. \( k = 1 \) i.e. \( a_1(z) = z \) and \( a_2(z) = \frac{az + b}{cz + d} \)

2. \( b = 0 \) i.e. \( a_1(z) = kz \) and \( a_2(z) = \frac{az}{cz + d} \)

3. \( c = 0 \) i.e. \( a_1(z) = kz \) and \( a_2(z) = \frac{az + b}{d} \)

Given any of the above three cases the mappings \( a_1 \) and \( a_2 \) share a common fixed point. This is impossible since it would require the initial circles to overlap, contradicting Theorem 4.201 which required the circles to form a tangent chain. The commutator must still be parabolic to ensure nesting image circles at the tangency points, hence \( \text{tr}(a_1a_2A_1A_2) \) must equal \(-2\).

**Definition 4.201 – Circle and Quasicircle Limit Sets** (Mumford et al, 2002)

Groups whose limit sets are circles/lines (or wholly contained within a circle/line) are called fuchsian. Limit sets which are loops (connected continuous curves) are called quasicircles and these limit sets look vaguely circular but crinkled up. A group of Möbius transformations whose limit set is a quasicircle is called a quasifuchsian group. The limit set of a quasifuchsian group is a simple connected curve.

![Figure 4.211 – Circle (left) and Quasicircle (right) limit sets](image)

Under the condition described in Theorem 4.201 given four circles in a tangent chain the resulting limit set will be either a circle or a quasicircle.

When using Theorem 4.201 we paired opposite circles, these circles can be of different radii with no symmetric properties as Theorem 4.201 requires only a tangent
chain of circles. Mumford et al (2002) began their exploration of connected 2\textsuperscript{nd} order circle pairing iteration limit sets by focusing on symmetric (about the origin (O)) circle placements. Mumford et al (2002) developed a ‘recipe’ to produce circle and quasicircle limit sets using such symmetric tangent circle placements. Let circles $C_{a_1}$ and $C_{a_2}$ lie on the imaginary axis with common radii and their centres equidistant from the origin. Let the centres of circles $C_{a_2}$ and $C_{a_2}$ lie on the real axis and be equidistant from the origin. Circles $C_{a_2}$ and $C_{a_2}$ will also have common radii. Let $k$ be the ratio of the radii of circles $C_{a_1}$ and $C_{a_2}$, i.e. a scaling factor. A placement of this form is shown in figure 4.212:

Note: $x$, $y$, $u$, $v$ and $k$ are real valued parameters.

Since circles $C_{a_2}$ and $C_{a_2}$ lie on the real axis we will begin by determining an appropriate circle pairing mapping. Thus far we have used a systematic approach of determining circle pairing mappings by using the formula stated in Chapter 1 §6 pg 35. This formula will not be used when pairing circles under this recipe. The formula
performs a translation to the origin then a scaling to the unit circle. Once scaled an
inversion is performed. The circle is once again scaled and translated to coincide with
its paired circle. Although such a mapping would pair circles $C_{a_2}$ and $C_{A_2}$ it will not
ensure the tangency points are mapped as required by Theorem 4.201. A different
approach is needed to ensure the tangency points are mapped appropriately.

By construction, the circles $C_{a_2}$ and $C_{A_2}$ are orthogonal to the unit circle. Since
our circles are symmetrically placed they form a subset of a $\theta$-Schottky group. A $\theta$-
Schottky group is based on a symmetric arrangement of circles all orthogonal to the unit
circle. Label the point of intersection of $C_{a_2}$ and the unit circle $Q$. The centre of $C_{a_2}$ can
be expressed as $\frac{1}{\cos(\theta)}$ where $\theta$ is the angle the line segment $OQ$ makes with the real
axis. Mumford et al (2002) developed a circle pairing mapping for two circles centred
on the real axis orthogonal to the unit circle, as shown below:

$$
\begin{bmatrix}
1 & \cos(\theta) \\
\cos(\theta) & 1
\end{bmatrix}
= 
\begin{bmatrix}
csc(\theta) & \cot(\theta) \\
\cot(\theta) & \csc(\theta)
\end{bmatrix}
$$

The above mapping pushes circles orthogonal to the unit circle along the real axis, in a
positive direction, into other circles orthogonal to the unit circle. Furthermore the
mapping $a_2$ has fixed points at +1 and -1 and will push points on the real axis from the
fixed point -1 to the fixed point +1. Mappings such as $a_2$ are termed by Mumford et al
(2002) **special stretch maps** where a special stretch map is of the form:

$$
\begin{bmatrix}
s & t \\
t & s
\end{bmatrix}
\text{ where } s, t \in \mathbb{R} \text{ and } |s|^2 - |t|^2 = 1
$$

Given the circle placement in figure 4.212 we can take the circle pairing mapping found
using knowledge of the $\theta$-Schottky group and replace the variable $\theta$. We can then
express the circle pairing mapping in terms of the centres and radii of the circle $C_{a_2}$ and $C_{a_2}$:

\[ \frac{1}{\cos(\theta)} = \frac{x}{y}, \quad \sin(\theta) = \frac{1}{x} \]

\[ \csc(\theta) = x, \quad \cot(\theta) = y \]

The mapping $a_2$ which will pair the outside of $C_{a_2}$ to the inside of $C_{a_1}$ is given by

\[ a_2 = \begin{pmatrix} x & y \\ y & x \end{pmatrix} \]

We require all Möbius transformations to be normalized hence the condition $x = \sqrt{1+y^2}$ must hold for mapping $a_2$.

Since the placement shown in figure 4.212 is symmetric about the origin we can use knowledge of the circle pairing mapping $a_2$ to define a circle pairing mapping for circles $C_{a_1}$ and $C_{a_1}$, namely map $a_1$. Circles $C_{a_1}$ and $C_{a_1}$ are not (necessarily) orthogonal to the unit circle, they are orthogonal to a circle centred at the origin of radii $k$. To transform circle $C_{a_2}$ into $C_{a_1}$ we would need to rotate $90^\circ$ in an anti-clockwise direction and scale by a factor of $k$, likewise for our circle pairing mapping. Hence $a_1$ is given by

\[ a_1 = \begin{pmatrix} u & ikv \\ -iv & u \end{pmatrix} \]

Once again we require our Möbius transformations to be in normalized form and hence the condition $u = \sqrt{1+v^2}$ must be satisfied.

We not only required normalized matrices but in order to form a continuous limit curve the four initial circles $C_{a_1}, C_{a_1}, C_{a_2}$ and $C_{a_2}$ must form a tangent chain. The above mappings do not automatically ensure tangency points are mapped according to Theorem 4.201, as we will now see a further condition will need to be placed on the variables $x, y, u, v$ and $k$. This condition must ensure $C_{a_1}$ is tangent to $C_{a_2}$, $C_{a_1}$ is tangent to $C_{a_2}$, etc. The property that two circles are tangent if the distance between
their centres is equal to the sum of their radii, which can be done by inspection of figure 4.212 or algebraically via Pythagoras’ Theorem, is used to form the following equality:

\[
\begin{vmatrix}
  iuk & x \\
  v & y
\end{vmatrix} = \frac{k + 1}{v}
\]

\[
\begin{vmatrix}
  iuk & x^2 \\
  v & y
\end{vmatrix} = \frac{k + 1}{v}
\]

given \(x, y, u, v\) and \(k \in \mathbb{R}\)

\[
\frac{u^2k^2}{v^2} + \frac{x^2}{y^2} = \frac{k^2}{v^2} + \frac{2k}{vy} + \frac{1}{y^2}
\]

which simplifies to

\[
\frac{1}{2} \left( k + \frac{1}{k} \right) = \frac{1}{y} v
\]

This equality incorporates not only the scaling factor \(k\) but variables \(u\) and \(v\) used in mappings \(a_1\) and \(a_2\) respectively. The relationship between \(x, y, u, v\) and \(k\) is equivalent to \(a_2a_1A_2A_1\) being parabolic thus ensuring the mapping \(a_2a_1A_2A_1\) fixes the tangency points of circles \(C_{a_1}\) and \(C_{a_2}\).

If the scaling factor \(k\) equals one then all four circles \(C_{a_1}, C_{a_2}, C_{A_1}\) and \(C_{A_2}\) are orthogonal to the unit circle. This placement will display four-fold rotational symmetry and all circles are of equal radii, which implies radii equal to 1. If all circles are of equal radii then \(v = y\) which implies \(x = u\) and forms the generator matrices

\[
a_1 = \begin{pmatrix} \sqrt{2} & i \\ -i & \sqrt{2} \end{pmatrix} \quad a_2 = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}.
\]

The point of tangency between circles \(C_{a_1}, C_{a_2}, C_{A_1}\) and \(C_{A_2}\) will be at \(\pm e^{\frac{i\pi}{4}}\) and the Schottky limit set will be a circle.

If \(k > 1\) the circles \(C_{a_1}\) and \(C_{A_1}\) expand and the fixed points move further apart whereas if \(k < 1\), \(C_{a_1}\) and \(C_{A_1}\) contract. Irrespective of the choice of \(k\) the circles \(C_{a_1}, C_{a_2}, C_{A_1}\) and \(C_{A_2}\) will still form a tangent chain with nested circles tangent at all levels of iteration and as such the resulting limit set is a connected curve. If \(k \neq 1\) a
quasicircle limit set will be formed. Furthermore these quasicircle limit sets will be connected with non-integer fractal dimension between one and two.

Mumford et al (2002) suggests that one way to create further Schottky group limit sets is to utilise the limit set we have already found. This can be done via conjugation, that is, we create a new image/limit set by viewing the original image from a different perspective as a “co-ordinate change on the Riemann sphere” (Mumford et al, 2002 pg 172).

Theorem 4.201 required opposite circles in a tangent chain placement to be paired together. We will now pair adjacent circles in a tangent chain placement. Under iteration the mappings must still satisfy certain conditions to ensure a continuous limit set is formed. The circles are paired in a different arrangement to that of Theorem 4.201 resulting in parabolic mappings being used.

**Definition 4.202 - The Modular Necklace Theorem** (Mumford et al, 2002)

A modular necklace is a tangent chain of four circles in which adjacent disks are paired by two transformations $a_1$ and $a_2$ as shown in figure 4.213, with tangent points mapped as follows:

\[ a_1(P) = R \quad S = \text{Fix}(a_1) \quad P = \text{Fix}(a_2a_1) \]
\[ a_2(R) = P \quad Q = \text{Fix}(a_2) \quad R = \text{Fix}(a_1a_2) \]

![Figure 4.213](image)

In order for image circles to shrink near the tangency points $S$ and $Q$ the generators $a_1$ and $a_2$ must be parabolic. The resulting image circles must also shrink at
points $P$ and $R$ resulting in the mappings $a_1a_2$ and $a_2a_1$ being parabolic. Since $a_1a_2$ and $a_2a_1$ are conjugate we will work with only mapping $a_1a_2$. Hence we require mappings $a_1$, $a_2$ and $a_1a_2$ to be parabolic as well as $\text{Fix}(a_1) \neq \text{Fix}(a_2)$.

Unlike Theorem 4.201, altering the size and placement of the circles in the tangent chain does not form a quasicircle limit set. Mumford et al (2002) states:

"the rigours imposed by specifying that the two generators and their product are all parabolic actually ‘freeze’ the group"

That is, if a tangent chain of four distinct non-overlapping Schottky circles are paired in such a way as to conform to Definition 4.202 the group formed using the generators $a_1$ and $a_2$ are, up to conjugation, ‘the same’. Furthermore if the conditions of Definition 4.202 are satisfied the mapping $a_1$ and $a_2$ are always conjugate to the ‘standard’ modular group generated by

$$a_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad a_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad a_1a_2 = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$$

Given any mappings $a_\alpha$ and $a_\beta$, with the properties expressed above, we can always find a Möbius transformation $M$ such that $Ma_\alpha M^{-1} = a_1$ and $Ma_\beta M^{-1} = a_2$. Mapping $a_1$ has the property $a_1(0) = 0$ and mapping $a_2$ has the property $a_2(\infty) = \infty$. The four tangency points $P$, $Q$, $R$ and $S$ always lie on a circle which is the limit set of the group. The limit set is the circle orthogonal to all circles in the placement.

At present we have only considered tangent chains with four tangency points. These tangent chains when paired in such a way as to satisfy Theorem 4.201 or Definition 4.202 can form connected limit sets. These limit sets will have either a circle or quasicircle structure. In the case of quasicircle limit sets formed under Theorem 4.201 the fractal dimension will be between one and two. In figure 4.211 (ii) a set of four tangent non-overlapping circles are paired according to Theorem 4.201 to form a quasicircle limit set. This particular circle placement does not display any symmetric properties and the circles are of different radii. One pair of opposite circles, although
disjoint, have circumferences which nearly touch. If this happened there would be a fifth point of tangency and as a result would impact the structure of the limit set of the 2\textsuperscript{nd} order ICPS. If the two circles, say $C_{a_1}$ and $C_{a_2}$, had a common point of tangency the conditions set out in Theorem 4.201 must still be adhered to. If the Möbius transformation which paired $C_{a_1}$ and $C_{a_2}$ is parabolic the image circles would nest at the point of tangency. In effect the quasicircle limit set would be ‘pinched’ together at the point of tangency of circles $C_{a_1}$ and $C_{a_2}$. Our definition of a quasicircle required the limit set to be a simple connected curve, hence an additional point of tangency would render the limit set not a quasicircle but the union of two quasicircles. The limit set will have a fractal dimension between one and two. Limit sets formed under Definition 4.202 are circles; hence the tangent chain can never ‘pinch’ together as a fifth point of tangency would never lie on the (unique orthogonal) circle limit set.

Examples have been given by Mumford et al (2002) which require symmetric placement of the initial circles. This work can be extended to many other tangent chains. One such example studied by Mumford et al (2002) is seen in the Apollonius Gasket (figure 4.215 (i) and (ii)). Figure 4.215 (i) has four touching circles (seen in orange), circle $C_{a_1}$ is a circle of infinite radius represented by the real axis and the placement has six points of tangency. The points of tangency are as follows and are shown in figure 4.214:

- circles $C_{a_1}$ and $C_{a_1}$ are tangent at the origin 0.
- circles $C_{a_1}$ and $C_{a_2}$ are tangent at point $P$.
- circles $C_{a_1}$ and $C_{a_2}$ are tangent at point $Q$.
- circles $C_{a_2}$ and $C_{a_2}$ are tangent at point $-i$.
- circles $C_{a_2}$ and $C_{a_2}$ are tangent at point $R$.
- circles $C_{a_2}$ and $C_{a_2}$ are tangent at point $S$.

The generating matrices used to construct the Apollonian Gasket are as follows:
\[
\begin{pmatrix}
1 & 0 \\
-2i & 1
\end{pmatrix}
= \begin{pmatrix} 1 - i & 1 \\ 1 & 1 + i \end{pmatrix}
\]

Figure 4.214

\( tr(a_1) \) and \( tr(a_2) \) both equal two hence the mappings \( a_1 \) and \( a_2 \) are parabolic, as with Theorem 4.201, \( tr(a_1 a_2 A_1 A_2) = -2 \). \( Fix(a_1) \) is the origin corresponding to the tangency point of \( C_{a_1} \) and \( C_{A_1} \) and \( Fix(a_2) \) is at \(-i\), the tangency point of \( C_{a_2} \) and \( C_{A_2} \).

The limit set of the Apollonian Gasket can be seen ‘glowing’ in figure 4.215 (i). The limit set alone can be seen in figure 4.215 (ii). The limit set contains infinitely many subsets which are non-fractal.

The Apollonian Gasket still requires opposite circles to be paired and tangency points mapped to ensure a connected limit set. There existed only four tangency points when dealing with placements described in Theorem 4.201. These conditions are still seen in the Apollonian Gasket yet these conditions do not ensure the limit set remains connected at the additional two points of tangency. Figure 4.216 shows that due to the
fact circle $C_{A_i}$ is enclosed by all other disks in the placement, the ‘beads’ will not lie on a single curve but a collection of curves. Hence, the limit set will be a collection of connected curves.

The non-fractal subsets seen in figure 4.215 are formed when combinations of the initial circles and the image circles form a tangent chain. After the first iteration there will be three image circles contained within all initial circles, without loss of generality consider circle $C_{A_2}$. Due to the fact that tangency points are mapped in continuous chains, circles $C_{a_1}$, $C_{A_1}$, $C_{A_2a_1}$ and $C_{A_2A_1}$ form a tangent chain, as shown in figure 4.216.

Other such tangent chains will exist in the three other initial circles. These tangent chains will permeate through all levels of iterations creating continuous curves within the limit set. When dealing with the tangent chain comprised of circles $C_{a_1}$, $C_{A_2a_1}$, $C_{A_1}$ and $C_{A_2A_1}$, Mumford et al (2002) noted that the mapping $A_2A_1a_2$ pairs the image circles $C_{A_2a_1}$ and $C_{A_2A_1}$. Similar mappings would pair other initial circles/image circles in the placement. Circles $C_{a_1}$, $C_{A_1}$, $C_{A_2a_1}$ and $C_{A_2A_1}$ and their associated circle pairing mappings conform to Definition 4.202. That is, the tangent chain containing circles $C_{a_1}$, $C_{A_1}$, $C_{A_2a_1}$ and $C_{A_2A_1}$ form a modular group generated by the mappings $a_1$ and $A_2A_1a_2$ with parabolic elements $a_1$, $A_2A_1a_2$ and $a_1A_2A_1a_2$. Since these modular groups appear at all levels of iteration the limit set contains infinitely many non-fractal subsets and has fractal dimension between one and two.
It has been noted throughout this chapter that there does not exist a unique mapping which pairs two circles. In fact, there are infinitely many circle pairing mappings which pair the same set of circles. We saw how when the initial circle placement formed a tangent chain and displayed $180^\circ$ symmetry Mumford et al (2002) was able to define a set of generators which not only paired the circles but paired them according to Theorem 4.201. These generators were expressed in terms of the initial circles centres and radii. Given an initial set of four circles in a tangent chain which do not display any symmetry we are able to conclude that the limit set can be non-fractal if all four tangency points lie on the unique orthogonal circle and the conditions of Definition 4.202 are satisfied. If the conditions of Definition 4.202 are not satisfied yet the initial circles are still paired using circle pairing mappings the limit sets could be a fractal dust or a connected limit set formed under Theorem 4.201. As we have seen, limit sets formed under Theorem 4.201 can be non-fractal, a quasicircle or the union of quasicircles. We do not at present have a systematic way of determining generators for non-symmetric placements. We know through the work of Mumford et al (2002) the criteria our mappings must obey yet we do not have a unique formula/approach of pairing these circles to ensure the criteria are met. Since we can pair circles in different orders it is also possible that more than one set of generators will produce limit sets of equivalent structure. An algebraic approach of determining such generators using information such as the initial circles centres and radii would greatly assist the study of limit sets formed under ICPSs. This concludes the review of the work of Mumford et al (2002) related to iterated circle pairing systems on circles.

Mumford et al (2002) extended the work of fractal producing iterative systems using the fundamental properties discovered through the iteration of circle pairing mappings. Mumford et al (2002) created further fractal limit sets by excluding the initial circle placement and generating the limit sets using a purely algebraic approach. Mumford et al (2002) were able to produce fractals with detailed connected limit sets and fractal dimensions between one and two. Since this purely algebraic approach does not require a defined set of non-overlapping initial circles it will not be discussed within this thesis.

Mathematicians such as Andreev (2006) have created software which allows for the iteration of Möbius transformation applied to circles as well as non-circular objects.
This software can be found at http://www.wiu.edu/users/fa101/java/Mobius2a/. All of the diagrams of circle pairing iteration limit sets displayed within this thesis were created, unless otherwise stated, using this software.
Chapter 5 Extensions of Iterated Möbius Transformations

§1 Complex Inversion (CI)

Chapter 2 looked at iterative systems of circle inversion maps in $\mathbb{R}^2$ on restricted domain applied to non-overlapping circle placements. Equivalent systems could be formed in $\mathbb{C}$ using mappings of the form $f(z) = \frac{r^2}{z - q} + q$ where $q \in \mathbb{C}$ is the centre of the circle and $r \in \mathbb{R}$ is its radius. These mappings are not Möbius transformations as they are anti-conformal. Chapter 4 looked at iterative systems in $\mathbb{C}$ where the iterative systems comprised of pairs of circle pairing Möbius transformations. In this section we will look at iterative systems in $\mathbb{C}$ using mappings of the form $f(z) = \frac{r^2}{z - q} + q$. We will call mappings of this form Complex Inversion (CI) maps, denoted $i_{C_k}^C$. The mappings $i_{C_k}^C$ can be thought of geometrically as inversion across the circle $C_k$, centre $q \in \mathbb{C}$ with radius $r \in \mathbb{R}$, followed by a reflection across the line parallel to the real axis passing through $q$ as seen in figure 5.101. CI mappings are Möbius transformations with associated matrix

$$
\begin{pmatrix}
q & r^2 - q^2 \\
1 & -q
\end{pmatrix}.
$$

![Figure 5.101](image-url)
CI maps are not equivalent to the inversion maps (Definition 1.502/Definition 1.503) discussed in Chapter 1. CI maps do however display many of the same properties as inversion maps. For example CI is a conformal mapping (Property 1.508). Under a CI mapping the exterior and the interior of the circle of inversion are interchanged (Property 1.503). A CI map is a contraction mapping on a set exterior to the circle of inversion (Property 1.504), more specifically for a set totally disjoint to the circle of inversion and CI mappings are involutions (Property 1.506). When inversion is performed a circle orthogonal to the circle of inversion is mapped to itself and any circle which passes through a point and its inverse point lies on a circle orthogonal to the circle of inversion (Properties 1.505 and 1.507). Under CI Properties 1.505 and 1.507 are only satisfied when the line segment joining the centre of the inversion circle and the centre of the orthogonal circle is parallel to the real axis.

Under CI a circle is mapped to a circle and furthermore the circle of inversion maps to itself. Due to the reflective component of the CI map, circumference points of the inversion circle are no longer fixed but are inverted to other points on the circumference. When circle $S$, with centre $c+id$ and radius $s$, is inverted, using CI, across inversion circle $C$, with centre $a+ib$ with radius $r$, the circle $S'$, with centre $u+iv$ and radius $t$, is formed. The radius and centre of circle $S'$ can be determined via:

$$u + iv = a + ib + \frac{r^2}{(c-a)^2 + (d-b)^2 - s^2}((c-a) - i(d-b))$$

$$t = \frac{r^2 s}{(c-a)^2 + (d-b)^2 - s^2}$$

Figure 5.102
As with circle inversion performed on non-concentric, non-overlapping circles in $\mathbb{R}^2$, given two non-concentric, non-overlapping circles in $\hat{\mathbb{C}}$, the centre of the newly inverted circle $S'$ is not the inverse of the centre of circle $S$. The centre of $S'$ will coincide with the image of the centre of circle $S$ if and only if circles $C$ and $S$ are concentric.

§2 Iterated Complex Inversion Systems (ICISs)/Restricted ICISs

We will now look at the limit sets formed via iteration of CI mappings.

Definition 5.201 - Iterated Complex Inversion System (ICIS)/ $n^{th}$ order ICIS

An Iterated Complex Inversion System (ICIS) is a finite collection $I^C = \{I^C_{C_k}\}_{k=1}^n$ of complex circle inversion mappings $I^C_{C_k} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$, where the $C_k$'s are distinct.

As previously seen in relation to ICISs/ISISs ICISs are not examples of IFSs. We can impose restrictions as before, they being we will only considered non-overlapping circle placements that is circles with disjoint interiors and at most one points of contact between the boundaries of the circles of inversion. We will also restrict the domain of our mappings $I^C_{C_k}$ such that the scaling factor $s$ of each mapping satisfies $0 \leq s \leq 1$. As previously shown in relation to restricted $n^{th}$ order ICISs, a restricted $n^{th}$ order ICIS will produce under iteration a unique limit set. Once again convergence can be determined in a manner analogous to that described for a restricted $n^{th}$ order ICIS.

The essential difference between the limit sets formed under a restricted $n^{th}$ order ICIS and a restricted $n^{th}$ order ICIS is that the latter will only produce limit sets which are a finite number of points or a fractal dust.

The following figures show the formation of nested circles under various restricted 2nd order ICISs. From inspection it can be seen that although a unique limit set is formed it does not always lie on the line segment, $l$, joining the centres of the circles of inversion.
Even where the circles of inversion are pairwise tangent the resulting limit set may not be the point of tangency.

Limit sets formed under restricted 2\textsuperscript{nd} order I\textsc{cis}s applied to circles which have a line segment \((l)\) parallel to the real axis will have limit sets analogous to those found using standard inversion in \(\hat{\mathbb{R}}^2\). If \(l\) is parallel to the imaginary axis the limit set will consist of two distinct points not on \(l\) but within the circles of inversion and on rays, \(N_1\) and \(N_2\). These rays emanate from the centres of the circles of inversion and extend outside the circles of inversion. If \(l\) is neither parallel to the real or imaginary axis the limit set consists of two points which lie within the circles of inversion yet do not lie on either \(l\) or \(N_1\) and \(N_2\). In all cases the resulting limit sets are non-fractal.
The limit set of a restricted $n^{th}$ order ICIS can be found via a modified version of either The Deterministic or Random Iteration Algorithm for a Restricted $n^{th}$ order ICIS. We have used a variation of The Random Inversion Algorithm for a Restricted $n^{th}$ order ICIS to construct all restricted $n^{th}$ order ICIS limit sets.

As seen in Chapter 2 we could use the restricted 2nd order ICIS limit sets to produce/construct higher order limit sets via Frame’s Algorithm (Chapter 2 §1). Assume we have three distinct non-overlapping pairwise tangent circles, we will have either five or six distinct limit points which are the union of the restricted 2nd order ICIS limit sets, as seen in figure 5.203. Note that there exist five limit points for the three circle placement if and only if a pair of circles in the placement have centres which form a line segment parallel to the real axis, as shown in figure 5.204:

![Figure 5.203](image1)

![Figure 5.204](image2)

When forming the limit set of a restricted 3rd order ICIS we noted that all restricted 2nd order ICIS limit points lie on the boundary of a containment triangle. Due to the reflection incorporated into CI mappings our restricted 2nd order ICIS limit points lie outside (or on the boundary) of this containment triangle. This results in limit points which do not lie on the unique orthogonal circle. By definition the limit set must contain all limit points and hence the unique orthogonal circle is not fixed under CI. Consequently under CI all restricted 3rd order ICIS limit sets are fractal dusts. In order to generate a limit set with fractal dimension between one and two the limit set must contain a non-fractal subset. No non-fractal limit set can be formed under a restricted $n^{th}$ order ICIS and as such all limit sets will have a fractal dust structure.
A further difference of limit sets formed under a restricted $n^{th}$ order ICIS is that rotation of the circle placement will not result in rotation of the limit set, as shown in figure 5.205.

![Figure 5.205](image)

Figure 5.205

Through experimentation it does appear that in certain scenarios the limit sets will have either a curved nature (figure 5.205 (ii)) or appear to have limit points contained within curves (figure 5.205(i)).
Chapter 6 Further Work

This thesis provides an overview of fractal producing iterative mapping systems on circles. Inversion, CI and circle pairing mappings all utilized the property that the exterior of a given circle is mapped into the interior of either itself or another circle. Of course, fractal producing iterative systems can be defined which do not require this condition. Mumford et al (2002) pursued this avenue by considering iterative systems of Möbius transformations that retained some but not necessarily all the properties of circle pairing mappings to produce fractal limit sets via a purely algebraic construction.

In our review of the work of Mumford et al (2002) the generators of the Schottky groups paired circles. Groups such as these are termed classical Schottky groups (Mumford et al, 2002 and Williams, 2009). A non-classical Schottky group would have generators which pairs non-overlapping closed curves, where the curve is not a circle. We have seen that classical Schottky groups on two generators can produce fractal limit set. Hilbert (2009) produced a limits set a using non-classical Schottky group, as shown in figure 6.001.

![Figure 6.001 (Hilbert, 2009)](image)

Thus far our iterative systems required a minimum of two circles in order to produce a limit set. A minimum of three circles were required to produce fractal limit set under circle inversion iteration and a minimum of four circles under circle pairing iteration. Burns (2008) has created fractal limit sets under iteration of a system of Möbius transformations using a single circle, the unit circle. Figure 6.002 uses ten mappings in its construction, these maps are composed with a Möbius transformation which maps the unit circle to itself, denoted by $U(z)$. The maps $U(z)$ are special stretch...
maps and results in the points on the circle being rotated as well as ‘stretched’. The transformations which evenly distribute the image circles around the circumference of the unit circle (Stage 1) will in subsequent iterations map image circles which have been ‘stretched’ by the composition of scaling/translation transformations with mapping $U(z)$. The amount of ‘stretching’ will be determined by the value of the parameters of $U(z)$. The limit set shown in figure 6.002 has a fractal dimension between one and two.

![Figure 6.002 (Burns, 2008)](image)

We will now discuss avenues for further research in relation to fractal producing iterative systems on circles.

§1 Classification Conjectures for Restricted $n^{th}$ order ISIS Limit Sets

In Chapter 3 we discussed the formation of limit sets under sphere inversion iteration. A further avenue for research would be to verify the validity of the following classification conjecture.

**Conjecture 6.101 - Classification Conjecture for Restricted $n^{th}$ order ISIS Limit Sets**

Let $\{S_k\}_{k=1}^n$ be $n \geq 3$ distinct non-overlapping spheres contained within the metric space $\hat{\mathbb{R}}^3$. The limit set $\mathcal{L}(\{S_k\}_{k=1}^n)$ formed under spherical inversion iteration will have one of the following forms:
1. A sphere.
2. A fractal dust contained within the boundary of a sphere.
3. A fractal dust contained within multiple spheres.
4. A fractal containing a non-fractal subset(s).

Note:

i. For \( n = 3 \) only cases 1 and 2 apply, as discussed in Chapter 3 §3.

ii. If the centres of all inversion spheres lie on a single plane then the limit set is equivalent to that of a restricted \( n^{th} \) order ICIS (Theorem 2.201).

Furthermore, this plane will contain the restricted \( n^{th} \) order ISIS limit set where the fractal dimension of the limit set will be strictly less than two.

A summary of the above classification conjectures can be seen in the following diagram:
§2 Overlapping Iterated Circle /Spherical Inversion Systems

Avenues for expansion in relation to overlapping iterated circle/spherical inversion systems are as follows:

- Frame has begun the study of overlapping circles. Criteria/restrictions which determined if or when an overlapping circle placement will produce a fractal limit set under circle inversion iteration needs to be found.
- Once criteria are defined for overlapping circle placements a classification theorem analogous to that for non-overlapping circles could be formed.
- The limit sets formed under iteration of overlapping circles would form a subset of those formed under iteration of overlapping spheres, hence further study of possible overlapping spherical inversion limit sets is required. As with the work of Frame et al (2002), in relation to overlapping circles, the angle of intersection between the overlapping spheres of inversion would require investigation.
- The use of restricted limit sets in relation to overlapping circles has been developed by Frame and Booth (2003), the use of restricted limit sets in overlapping spherical inversion may provide further three dimensional fractal limit sets.
- In Chapter 5 §1 the idea of CI was introduced, an exploration of limit sets formed under iteration of complex circle inversion mappings on overlapping circles may also provide varying fractal limit sets.

§3 Limit Sets of Schottky Groups

Mumford et al (2002) provided us with an extensive and detailed introduction to the ideas and processes used to iterate circle pairing Möbius transformations. The work of Mumford et al (2002) led us to pose the following questions and consider the following avenues for further research:

- For 2\textsuperscript{nd} order ICPSs we saw how symmetric placements aided the creation of a ‘recipe’ to form fractal limit sets. By looking at arbitrary placements in the plane...
which display no symmetric properties are we able to form fractal sets by defining conditions of the circle pairing mappings?

- The Apollonian Gasket showed a four circle tangent chain with symmetric properties. By looking at placements which display a high level of symmetry can further fractal limit sets be formed?
- When studying restricted $n^{th}$ order ICIS limit sets we found limit sets which had both fractal and non-fractal subsets. Can such limit sets be formed under an $n^{th}$ order ICPS. In order to achieve this we would need to begin further study into higher order ICPSs. We can see in figure 6.301 the existence of a higher order ICPS (3$^{rd}$ order ICPS) which has a fractal limit set.

Given a circle placement with say six distinct circles. Let a set of four circles in the placement form tangent chain with the remaining two circle totally disjoint to all other circles. We know we can construct a 2$^{nd}$ order ICPS which will produce a non-fractal subset using a tangent chain, would such a placement produce a limit set which is disconnected with non-fractal subsets?

- Can fractal limit sets be produced using an even number of generators yet an odd number of initial circles? Can a fractal limit set be formed using only three circles? Can we create a fractal limit set using six Möbius transformations and only three initial circles paired as shown in figure 6.302? If fractal limit sets can be produced using such placements what structure(s) will the limit set exhibit?
Throughout the text *Indra’s Pearls: The Vision of Felix Klein* the initial circles were either non-overlapping or tangent. Is it possible to produce fractal limit sets using overlapping initial circles? Mumford et al (2002) dropped the condition that initial circles must be disjoint or tangent to create **Schottky crossing groups**. They showed that if the angle of intersection between the circles is $45^\circ$ a non-fractal limit set emerges (figure 6.303 (i)). They also showed an example of a quasicircle limit set in figure 6.303 (ii). Further expansion into these Schottky crossing groups may yield many detailed fractal images.
Appendices:

Maple Algorithms

The following is an algorithm which will produce an approximation to the circle inversion limit set on \( n \) distinct circles. This algorithm is used for non-overlapping circles yet can be applied to all circle placements. The restriction that when choosing a circle at random from the \( C_i \)’s, say \( C_n \), that \( C_m \neq C_n \) is not enforced since under circle inversion iteration this is an involution and will not impact the resulting limit set. In the case of overlapping circles this algorithm does not apply any exclusions to restrict the limit sets as discussed in Chapter 3 §1.

\[
\text{restart:}
\]
\[
\text{with(plots):}
\]
\[
\text{Warning, the name changecoords has been redefined}
\]

Procedure for inverting a point in a circle

\[
\text{invertpt:=proc(x,y,a,b,r)}
\]
\[
\text{local ix, iy:}
\]
\[
\text{ix:=(a)+((r^2)/(((x-a)^2)+((y-b)^2))*(x-a))}
\]
\[
\text{iy:=(b)+((r^2)/(((x-a)^2)+((y-b)^2))*(y-b))}
\]
\[
\text{[ix, iy]:}
\]
\[
\text{end:}
\]

User inputted data, pt1 is chosen arbitrarily

\[
\text{pt1:=[a,b]:}
\]
\[
\text{a1:=a1: b1:=b1: r1:=r1:}
\]
\[
\text{a2:=a2: b2:=b2: r2:=r2:}
\]
\[
\text{a3:=a3: b3:=b3: r3:=r3:}
\]
\[
\text{a4:=a4: b4:=b4: r4:=r4:}
\]
\[
\vdots
\]
\[
\text{an:=an: bn:=bn: rn:=rn:}
\]

Procedure for generating points on a circle

\[
\text{sircle:=proc(a,b,r)}
\]
\[
\text{local i, x, y, data:}
\]
\[
\text{global Npts:}
\]
\[
\text{data:=[],}
\]
\[
\text{for i from 1 to Npts do}
\]
\[
\text{x:=a+r*cos(i*2*Pi/Npts):}
\]
\[
\text{y:=b+r*sin(i*2*Pi/Npts):}
\]
\[
\text{data:=[op(data),[x,y]]:}
\]
od:
end:

Select # of points - between 1000-5000 points is sufficient

Npts:=1000:

Generating the $n$ inversion circles

data1:=sircle(a1,b1,r1):
data2:=sircle(a2,b2,r2):
data3:=sircle(a3,b3,r3):
data4:=sircle(a4,b4,r4):

$\vdots$

data:n:=sircle(an,bn,rn):
plt1:=plot(data1,color=green):
plt2:=plot(data2,color=green):
plt3:=plot(data3,color=green):
plt4:=plot(data4,color=green):

$\vdots$

pltn:=plot(data:n,color=green):

Starting with any point outside the circles, i.e. pt1, randomly invert in the other circles

Note: datad in the following procedure has the property that in the above section the letter d has not been used to label a circle.

die:=rand(1..n):
datad:=[pt1]:
x:=op(1,pt1): y:=op(2,pt1):
for i from 1 to Npts do
    if die()=1 then
        pt:=invertpt(x,y,a1,b1,r1):
    elif die()=2 then
        pt:=invertpt(x,y,a2,b2,r2):
    elif die()=3 then
        pt:=invertpt(x,y,a3,b3,r3):
    elif die()=4 then
        pt:=invertpt(x,y,a4,b4,r4):
    $\vdots$

    elif die()=n then
        pt:=invertpt(x,y,an,bn,rn):
    end if:
    x:=op(1,pt): y:=op(2,pt):
    if i>50 then
        datad:=[op(datad),pt]:
    end if:
end if:
pltd:=plot(datad,style=POINT,color=red):
display(plt1,plt2,plt3,plt4,...,pltn,pltd);
To produce equivalent limit sets in the extended complex plane the following algorithm can be used.

```plaintext
code
restart:
with(plots):
Warning, the name changecoords has been redefined

Procedure for inverting a point in a circle
invertpt:=proc(z,a,b,r)
local w:
w:=((r^2)/(conjugate(z-(a+b*I))))+(a+b*I);
end:

User inputted data, pt1 is chosen arbitrarily
pt1:=[a,b]:
a1:=a1: b1:=b1: r1:=r1:
a2:=a2: b2:=b2: r2:=r2:
a3:=a3: b3:=b3: r3:=r3:
a4:=a4: b4:=b4: r4:=r4:
:
an:=an: bn:=bn: rn:=rn:

Select # of points - between 1000-5000 points is sufficient
Npts:=1000:

Procedure for generating points on a circle
sircle:=proc(a,b,r)
local i,z,data:
global Npts:
data:=[ ]:
for i from 1 to Npts do
z:=(a+r*cos(i*2*Pi/Npts))+I*( b+r*sin(i*2*Pi/Npts));
data:=[op(data),z]:
end:
end:

Generating the n inversion circles
data1:=sircle(a1,b1,r1):
data2:=sircle(a2,b2,r2):
data3:=sircle(a3,b3,r3):
data4:=sircle(a4,b4,r4):
:
datan:=sircle(an,bn,rn):
plt1:= complexplot(data1,color=green):
plt2:= complexplot(data2,color=green):
plt3:= complexplot(data3,color=green):
plt4:= complexplot(data4,color=green):
:
pltn:= complexplot(datan,color=green):
```

Starting with any point outside the circles, i.e. pt1, randomly invert in the other circles

**Note:** datad in the following procedure has the property that in the above section the letter d has not been used to label a circle.

```plaintext
pt1:=0:
die:=rand(1..n):
datad:=[pt1]:
z:= pt1:
for i from 1 to Npts do
    if die()=1 then
        pt:=invertpt(z,a1,b1,r1):
    elif die()=2 then
        pt:=invertpt(z,a2,b2,r2):
    elif die()=3 then
        pt:=invertpt(z,a3,b3,r3):
    
    elif die()=n then
        pt:=invertpt(z,an,bn,rn):
    end if:
    z:=pt:
    if i>50 then
        datad:=[op(datad),pt]:
    end if:
end:
pltd:=complexplot(datad,style=POINT,color=red):
display(plt1,plt2,plt3,plt4,...,pltn,pltd);
```

**Proofs - Chapter 1**

**Proofs from § 2**

**Theorem 1.201 - Contraction Mapping Principle** (Hart, 1996)

Let \((X, d)\) be a complete metric space and let \(w: X \rightarrow X\) be a contraction on \(X\). Then \(w\) has a unique fixed point \(x_0\) in \(X\).

**Proof:**

To show that at least one fixed point exists, pick any point \(x_0 \in X\) and iterate \(w\) on it, creating the orbit
The distances between successive points in the orbit can be characterized as
\[ d(x_{i+1}, x_i) \leq sd(x_i, x_{i-1}) \]
where \( s = \text{Lip}(w) \) (Lipschitz constant). This gives, by induction,
\[ d(x_{i+1}, x_i) \leq s^i d(x_1, x_0) \]
Thus, \( \{x_i\} \) is a Cauchy sequence and converges in a complete metric space. Let \( x \in X \) be the limit point of the sequence. The point \( x \) is fixed under \( w \) since
\[
\begin{align*}
  w(x) &= w(\lim_{j \to \infty} x_j) \\
   &= \lim_{j \to \infty} x_{j+1} \\
   &= x
\end{align*}
\]
Now, suppose there were at least two distinctive fixed points \( X \subseteq Y \). Then
\[ d(w(x), w(y)) = d(x, y) \]
By Definition 1.202, the map \( w \) cannot be a contraction. Thus, there can only be one fixed point of \( w \).

\[ \square \]

**Corollary 1.201** (Hart, 1996)

Let \( w, X \) be as in Theorem 1.201. Then all points in \( X \) converge to the fixed point under iteration of \( w \).

\[ \text{Proof:} \]

Since the \( x \) in Theorem 1.201 was chosen arbitrarily all points in \( X \) must converge to the fixed point of \( w \). The proof of Theorem 1.201 also shows that there exists only one fixed point of \( w \).
Proof of The Cantor Middle Third Set

Cantor Set (Wachsmuth, 2000):

A non-empty set X is a Cantor set if it satisfies the following properties:

1. The set contains no open sets.
2. The set is compact.
3. The set has zero length.
4. The set is perfect.

Proof:

1. The Cantor Middle Third Set does not contain any open sets.

Let $A_0 = [0,1]$ and define, for each $n$, the sets $A_n$ recursively as

$$A_n = A_{n-1} \setminus \bigcup_{k=0}^{\infty} \left( \frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right).$$

Then The Cantor Middle Third Set is given as

$$C = \bigcap A_n.$$  Another way to write The Cantor Middle Third Set is to note that each of the sets $A_n$ can be written as a finite union of $2^n$ closed intervals, each of which has a length of $\frac{1}{3^n}$, as follows:

$$A_0 = [0,1]$$
$$A_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$$
$$A_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{3}{9}] \cup [\frac{6}{9},\frac{7}{9}] \cup [\frac{8}{9},1]$$

etc.

Now suppose that there is an open set $U$ contained in $C$. Then there must be an open interval $(a,b)$ contained in $C$. Now pick an integer $N$ such that $\frac{1}{3^N} < b - a$. Then the interval $(a,b)$ cannot be contained in the set $A_N$, because that set is comprised of
intervals of length $\frac{1}{3^n}$. However, if that interval is not contained in $A_n$, it cannot be contained in C. Hence, no open set can be contained in The Cantor Middle Third Set $C$.

2. The Cantor Middle Third Set is Compact.

Let $A_0 = [0,1]$ and define, for each $n$, the sets $A_n$ recursively as

$$A_n = A_{n-1} \setminus \bigcup_{k=0}^{\infty} \left( \frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right).$$

Then The Cantor Middle Third Set is given as

$$C = \bigcap A_n.$$ Each set $\bigcup_{k=0}^{\infty} \left( \frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right)$ is open. Since $A_0$ is closed, the set $A_n$ are all closed as well, which can be shown by induction. In addition, each set $A_n$ is a subset of $A_0$, so that all sets $A_n$ are bounded.

Hence, $C$ is the intersection of closed, bounded sets, and therefore $C$ is closed and bounded. Then $C$ is compact.

3. The Cantor Middle Third Set has zero length, but contains uncountable many points.

Let $A_0 = [0,1]$ and define, for each $n$, the sets $A_n$ recursively as

$$A_n = A_{n-1} \setminus \bigcup_{k=0}^{\infty} \left( \frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right).$$

Then The Cantor Middle Third Set is given as

$$C = \bigcap A_n.$$ To be more specific, we have:

$$A_0 = [0,1]$$

$$A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$

$$A_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$$

$$\vdots$$

$$\text{etc.}$$

That is, at the $n^{th}$ stage ($n>0$) we remove $2^{n-1}$ intervals from each previous set, each having length $\frac{1}{3^n}$. Therefore, we will have a total length of
\[ \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} \]
\[ = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n} \]
\[ = \frac{1}{3} \left(\frac{1}{1-2/3}\right) = 1 \]

from the unit interval \([0,1]\). Since we remove a set of total length one from the unit interval, the length of the remaining Cantor Middle Third Set must be zero.

The Cantor Middle Third Set contains uncountable many points because it is a perfect set.

4. The Cantor Middle Third Set is Perfect and hence uncountable.

Let \( A_0 = [0,1] \) and define, for each \( n \), the sets \( A_n \) recursively as
\[
A_n = A_{n-1} \setminus \bigcup_{k=0}^{\infty} \left( \frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right). \]
Then The Cantor Middle Third Set is given as
\[
C = \bigcap A_n. \]

From this representation, it is clear that \( C \) is closed. Next, we need to show that every point in The Cantor Middle Third Set is a limit point.

One way to do this is to note that each of the sets \( A_n \) can be written as a finite union of \( 2^n \) closed intervals, each of which has a length of \( \frac{1}{3^n} \), as follows:

\[
A_0 = [0,1] \\
A_1 = [0,\frac{1}{3}] \cup \left[\frac{2}{3},1\right] \\
A_2 = [0,\frac{1}{9}] \cup \left[\frac{2}{9},\frac{3}{9}\right] \cup \left[\frac{6}{9},\frac{7}{9}\right] \cup \left[\frac{8}{9},1\right] \\
\vdots \\
\text{etc.} 
\]

Note that all endpoints of every subinterval will be contained in The Cantor Middle Third Set. Now take any \( x \in C = \bigcap A_n \). Then \( x \) is in \( A_n \) for all \( n \). If \( x \) is in \( A_n \), then \( x \) must be contained in one of the \( 2^n \) intervals that comprise the set \( A_n \). Define \( x_n \)
to be the left endpoint of that subinterval (if $x$ is equal to that endpoint, then let $x_n$ be
equal to the right endpoint of that subinterval). Since each subinterval has length $\frac{1}{3^n}$,
we have:

$$|x - x_n| < \frac{1}{3^n}$$

Hence, the sequence $\{x_n\}$ converges to $x$, and since all endpoints of the
subintervals are contained in The Cantor Middle Third Set, we have found a sequence
of numbers contained in $C$ that converges to $x$. Therefore, $x$ is a limit point of $C$. Since $x$
was arbitrary, every point of $C$ is a limit point. Since $C$ is also closed, it is then perfect.

Since every perfect set is uncountable, so is The Cantor Middle Third Set.

**Radical Centre/Orthogonal Circles (2D)**

**Constructing an orthogonal circle through three given circles:**

Given three non-overlapping circles, label them $C_1$, $C_2$ and $C_3$ and express them
algebraically:

$$\begin{align*}
(x - a)^2 + (y - b)^2 &= r_1 \\
(x - c)^2 + (y - d)^2 &= r_2 \\
(x - e)^2 + (y - f)^2 &= r_3
\end{align*}$$

\[\text{s.t. } (a,b) \neq (c,d) \neq (e,f) \text{ and } r_i > 0 \; \forall r_i\]

1. **Construct the Radical Axis for each pair of Circles.**

Constructing the Radical Axis for the pair of circles (1) and (2), (2) and (3) and
(1) and (3) requires the following construction.

Property 1.505/1.507 gives a means of creating a circle orthogonal to each pair
of circles. Without loss of generality choose the pair of circles (1) and (2). Choose a
point $x$ not on the circumference of either circle and invert this point into each circle to
create inverse points $x_1$ and $x_2$. Property 1.501 states that a circle can be formed which
passes through all three of these points and this circle will be orthogonal to both circles (1) and (2). The Radical Axis for the pair of circles (1) and (2) will pass through the centre of the orthogonal circle and be perpendicular to the line which passes through the points (a, b) and (c, d). Proof of this relies on the following definitions:

**Note:** If the circles were concentric then no orthogonal circle can be formed.

**Definition A1 - Power of a Point** (King, 2004)

Given a circle \( C \) with centre \( O \) and radius \( r \), for any point \( A \) there is a number called the Power of a Point \( A \) with respect to the circle \( C \).

\[
\text{The Power of a Point with respect to } C = |OA|^2 \times r^2 .
\]

**Definition A2 - Radical Axis** (King, 2004)

The Radical Axis of two circles is the line that contains the centre points for all circles that are orthogonal to both of the given circles. If the given circles do not intersect, then all of the orthogonal circles that are centred on the Radical Axis intersect each other at the same two points. The Radical Axis of intersecting circles is their common secant, and the orthogonal circles do not intersect each other.

The definition of the Radical Axis can be restated to incorporate the property of The Power of a Point.

**Property A1 - Radical Axis** (King, 2004)

Given two circles \( C_1 \) and \( C_2 \), the set of points \( A \) for which the power of \( A \) with respect to \( C_1 \) and \( C_2 \) are equal is called The Radical Axis of \( C_1 \) and \( C_2 \).

Given that the arrangement of circles chosen are non-concentric the Radical Axis will form a line. This line is termed the Radical Line, this line is defined as:
Property A2 - The Radical Line (King, 2004)

The locus of points of equal circle power with respect to two non-concentric circles which is perpendicular to the line of centres.

Monge’s Problem (Dörrie, 1965) shows that the radical axis is the locus of the centres of all the circles that are perpendicular to the initial two circles.

Label the newly constructed circle \( \Sigma \) with centre \( \Omega = (\alpha, \beta) \) and radius \( \rho \).

Construct a line through the centres of circles (1) and (2) and label it \( L \). Let \( A = (a, b) \) and \( B = (c, d) \). Let the lines \( A\Omega \) and \( B\Omega \) be equal to the lengths of \( u \) and \( v \) respectively. Then

\[
2^2 \rho^2 = u^2 - v^2 - r_1^2 \quad \text{............... (4)}
\]

since by the Pythagorean Theorem they are both equal to \( \rho^2 \) the radius of circle \( \Sigma \). This implies that circles (1) and (2) have the same power at \( X \). The locus of the point \( X \) at which the two circles (1) and (2) have the same power needs to be constructed. If \( X \) is a point possessing this locus and the perpendicular from \( \Omega \) intercepts the line \( L \) at the point \( G \), and, moreover, if \( AG = m \) and \( BG = n \), then the Pythagorean Theorem states that the square of the perpendicular is equal to

\[
u^2 - m^2 = v^2 - n^2 \quad \text{............... (5)}
\]

Solving equations (4) and (5) results in

\[
m^2 - r_1^2 = n^2 - r_2^2 \quad \text{............... (6)}
\]
implying that circles (1) and (2) possess equal powers at $G$. Given that the distances $m$ and $n$ are always positive and that $m + n = L$ the previous equation gives fixed values for $m$ and $n$. Consequently every locus point $X$ must lie on the line perpendicular to $L$ and which passes through point $X$.

This line is the Radical Axis for circles (1) and (2).

2. **Construct the Radical Centre for Three Circles.**

Given three Radical Axes, label them appropriately:

$$y_{(1,2)} = \frac{a - c}{b - d} x - q_1 + \frac{a - c}{b - d} q_2$$

$$y_{(1,3)} = \frac{a - e}{b - f} x - w_1 + \frac{a - e}{b - f} w_2$$

$$y_{(2,3)} = \frac{c - e}{d - f} x - z_1 + \frac{c - e}{d - f} z_2 \hspace{1cm} \ldots \ldots \ldots \ldots (7)$$

where $(q_1, q_2), (w_1, w_2)$ and $(z_1, z_2)$ are the centres of the orthogonal circles described above.

Monge’s Problem states that the three Radical Lines intersect at one point, this point being the Radical Centre, otherwise known as the power centre of the three circles.

**Theorem A1 - Theorem of Monge** (Dörrie, 1965)

The three radical axes of three circles pass through a point known as the power centre of the three circles.

**Proof:**

Let the circles be (1), (2) and (3). We determine the point of intersection $O$ of the radical axes of the two pairs ((2),(3)) and ((1) ,(3)). At this point:

a) (2) and (3)

b) (1) and (3)
possess equal powers; consequently (2) and (1) have the same power at \( O \), i.e. \( O \) lies on the radical axis of (1) and (2).

Hence, by solving two of the three equations in (7) simultaneously the point \( O = (a_1, a_2) \) is found. This point is the Radical Centre of circles (1), (2) and (3), as seen in figure A2.

![Figure A2: Radical Centre](image)

3. **Construct the Orthogonal Circle.**

To complete the construction of the orthogonal circle the circles radius must be defined. The radius \( r_0 \) is found by applying Pythagoras’ theorem. Firstly, find the distance between the Radical Centre \( (O) \) and the centre of one of the three given circles, without loss of generality choose circle (1).

\[
d(O, (a, b)) = d_i = r_i + r_0^2
\]

Since circle (1) has radius \( r_i \) Pythagoras can be applied as follows:

\[
d_i^2 = r_i^2 + r_0^2
\]

\[
\Rightarrow r_0 = \sqrt{d_i^2 - r_i^2}
\]

Resulting in the equation of the orthogonal circle being,

\[
(x - a_1)^2 + (y - a_2)^2 = r_0
\]

Since the circles chosen are non-overlapping and non-concentric the orthogonal circles will always exist, as seen in figure A3.
If any pair of Radical Axes are parallel they fail to intersect, in this case the Radical Axes are said to intersect at the point of infinity. If this occurs then the orthogonal circle is the circle with infinite radius with centre at the point of infinity, i.e. the line that passes through the centre of all three circles.

**Power of a Point/Radical Centre/Orthogonal Spheres (3D)**

**Definition A3 - Power of a Point** (Boehm and Prautzsch, 1994)

In \( \mathbb{R}^3 \) a sphere is given in Cartesian co-ordinates by its normalized equation

\[
S(x) = x'x - 2m'x + c = 0
\]

One can compute the sphere’s intersection with a straight line \( L \) given by \( x = p + \nu \lambda \)

where \( \nu' \nu = 1 \), to obtain a quadratic equation in terms of \( \lambda \),

\[
\alpha \lambda^2 + \beta \lambda + \gamma = 0
\]

where \( \alpha = \nu' \nu = 1 \) and \( \gamma = S(p) \). Let \( \lambda_1 \) and \( \lambda_2 \) denotes the equation’s roots; then by Vieta’s formula \( \lambda_1 \lambda_2 = \gamma \). The number \( \gamma = S(p) \) is called the power of \( p \) with respect to \( S \). The power has the following geometric interpretations.

Since \( \nu' \nu = 1 \) the values \( \lambda \) form a metric scale on \( L \), i.e. one has \( \lambda = |x - p| \).

Thus \( \lambda_1 \lambda_2 = \gamma \) is the product of the distances between each of the two intersection points and \( p \). This product does not depend on \( \nu \). In particular, if \( L \) is tangent to \( S \), one
has \( \lambda_1 = \lambda_2 \). Therefore \( \sqrt{\gamma} = \sqrt{\lambda_1 \lambda_2} \) is the distance between \( p \) and the point of contact of any tangent of \( S \) containing \( p \). Note that \( p \) lies on \( S \) if \( \gamma = 0 \), and \( p \) lies inside \( S \) if \( \gamma < 0 \).

Figure A4 : Power of a point (circle and sphere) (Boehm and Prautzsch, 1994)

**Definition A4 - Vieta’s Formula** (PlanetMath.org, 2009)

Suppose \( P(x) \) is a polynomial of degree \( n \) with roots \( r_1, r_2, \ldots, r_n \) (not necessarily distinct). For \( 1 \leq k \leq n \), define \( S_k \) by

\[
S_k = \sum_{1 \leq a_1 < a_2 < \ldots < a_k \leq n} r_{a_1} r_{a_2} \ldots r_{a_k}
\]

The writing \( P(x) \) as

\[
P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0
\]

we find that

\[
S_i = (-1)^i \frac{a_{n-i}}{a_n}
\]

If \( P(x) \) is a polynomial of degree 2, then

\[
P(x) = a_2 x^2 + a_1 x + a_0
\]

and

\[
r_1 + r_2 = -\frac{a_1}{a_2}
\]

and

\[
r_1 r_2 = \frac{a_0}{a_2}.
\]
More interestingly, if \( P(x) = a_3x^3 + a_2x^2 + a_1x + a_0 \), then

\[
\begin{align*}
r_1 + r_2 + r_3 &= \frac{a_2}{a_3} \\
r_1r_2 + r_2r_3 + r_3r_1 &= \frac{a_1}{a_3}
\end{align*}
\]

and

\[
r_1r_2r_3 = \frac{a_0}{a_3}.
\]

**Definition A5 - Radical Centre** (Boehm and Prautzsch, 1994)

The powers of a point \( x \) with respect to two spheres \( S_1 \) and \( S_2 \) are equal if:

\[
x' x - 2m_1'x + c_1 = x' x - 2m_2'x + c_2,
\]

i.e. if \( x \) lies in a plane \( P_{12} \) defined by

\[
2[m_2 - m_1]'x - (c_2 - c_1) = 0.
\]

This plane is called the Radical Plane of both spheres or the Radical Axis if the spheres are circles. Note that this plane contains the real or non-real (complex) intersection of both spheres, where both powers vanish, and that it is perpendicular to the line spanned by the midpoints of both spheres.

For the three Radical Planes of any pair of the three spheres \( S_1, S_2, S_3 \) one has \( P_{12} + P_{23} + P_{31} = 0 \), i.e. the three planes either meet in a line or are parallel.

It follows that in \( \mathbb{R}^3 \) the six Radical Planes corresponding to four spheres meet at a point which is called the Radical Centre of the four spheres.

Analogously, in \( \mathbb{R}^2 \) the three Radical Axes of three circles meet at a point, the Radical Centre of the three circles, as illustrated in figure A5. Note that this configuration can be viewed as the intersection of a plane with three spheres and their Radical Planes in \( \mathbb{R}^3 \).
Definition A6 - Orthogonal Spheres (Boehm and Prautzsch, 1994)

The geometric meaning of the power $S(p)$ implies that each sphere around $p$ with radius $\sqrt{S(p)}$ meets $S$ orthogonally. Consequently, the midpoint of any sphere which meets two spheres $S_1$ and $S_2$ orthogonally lies in the radical plane of $S_1$ and $S_2$. There exists exactly one sphere orthogonal to four spheres given in general position in $\mathbb{R}^3$. Its midpoint $m$ is their Radical Centre; its radius is the root of the power of $m$.

Analogous properties hold for orthogonal circles in the plane, as shown in figure A6.

The radii $r_1$ and $r_2$ of two orthogonal spheres and the distance $d$ of their midpoints are related by

$$d^2 = r_1^2 + r_2^2$$
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